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THE MATHEMATICAL THEORY

OF

ELECTRICITY AND MAGNETISM

WATSON AND BURBURY

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OF

ELECTRICITY AND MAGNETISM

BY

H. W. WATSON, D.Sc., F.R.S.
FORMERLY FELLOW OF TRINITY COLLEGE, CAMBRIDGE

AND

S. H. BURBURY, M.A.
FORMERLY FELLOW OF ST. JOHN'S COLLEGE
CAMBRIDGE

VOL. II

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PREFACE.

THIS volume is mainly concerned with the application of electrical theory to current phaenomena, especially in their magnetic manifestations.

The subject has been greatly developed mathematically and experimentally in the last few years; but while much additional insight has been gained into the relations between them, the intrinsic nature both of electricity and magnetism remains yet to be discovered.

As stated in the preface to our first volume, the electric fluids cannot be regarded as physical realities, although they are most useful as the basis of a theory accounting for and to some extent predicting electrical phaenomena. And as regards the magnetic fluids, it may be doubted whether their existence would have been conceived at all if the order of discovery had been inverted and the magnetic properties of electric currents had become known to us before, instead of after, those of the loadstone and so-called permanent magnets. Not that the Ampère theory of the electromagnetic constitution of natural magnets would have been sufficient, inasmuch as it fails to include and explain many of the phaenomena of induced magnetism.

In this volume we have proceeded on the lines laid

down by Maxwell, adopting his conception of displacement and displacement currents, but not so as to exclude reference to other theories.

According to this displacement hypothesis of Maxwell, adopted in a modified form by Helmholtz also, there is a wave propagation of electric disturbance through different media with a velocity depending upon certain measurable electric and magnetic properties of the media, and it is found that the velocity as so determined agrees, within no wide limits, with the velocity of light in the respective media. Hence an electromagnetic theory of light has been propounded, of great beauty and simplicity, and free from some of the difficulties attaching to the older undulatory theory founded on the wave propagation of disturbance through an elastic luminiferous ether.

Until very recently, however, this electric disturbance propagation was hypothetical only and fortified by no independent experimental evidence; but within the last two years the researches of Hertz in Germany, based upon experiments with rapidly oscillating charges of electricity in finite conductors, experiments which have been reproduced and developed by Professors Fitzgerald, Lodge, and others in Great Britain, have supplied independent and almost demonstrative evidence of the existence of this disturbance propagation, and thus have invested the Maxwellian hypothesis with great additional interest.

We trust that the importance of certain portions of our subject and the advantage of considering them under different aspects may excuse the detail with

which they have been treated; this remark applies especially to the interesting but difficult investigation of induced currents in sheets and solids treated of in Chaps. XXII and XXIII.

In this investigation we have restricted our examples to such as would serve to illustrate general principles without involving too much analytical complexity, indicating memoirs and papers in which special cases requiring more elaborate mathematical treatment have been considered.

In Chap. II, Art. 19 of our first volume, at the bottom of page 21, there is an error in sign in the fundamental definition of differentiation with regard to an axis; this error is repeated again in Art. 25, and leads to the omission of the sign factor $(-1)^r$ in the expression for zonal spherical harmonics, we desire therefore to notice and correct it.

We also desire to acknowledge a correction by Dr. J. Nieuwenhuyzen Kruseman, who has pointed out an error in the latter part of Art. 141. See his very interesting memoir 'On the potential of the electric field in the neighbourhood of a spherical bowl charged or under influence' (*Phil. Mag.*, July, 1887).

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According to this convention the integral $\int x\,dy$ taken round a closed curve in the plane of xy is positive, and $\int y\,dx$ is negative. Similarly $\int z\,dx$ is positive, and $\int x\,dz$ negative, $\int y\,dz$ is positive, and $\int z\,dy$ negative.

If $d\sigma$ be any elementary plane area, l, m, n the direction-cosines of its normal, we have, taking the integrals round its boundary,

$$\int x\,dy = +n\,d\sigma,$$
$$\int y\,dx = -n\,d\sigma,$$
$$\int z\,dx = +m\,d\sigma,$$
$$\int x\,dz = -m\,d\sigma,$$
$$\int y\,dz = +l\,d\sigma,$$
$$\int z\,dy = -l\,d\sigma.$$

Let us next consider a curved surface bounded by a closed curve or curves. It can be divided into an infinite number of elementary plane areas. Let us choose the positive side of any one of these. If the surface does not cut itself we thereby determine the positive side of every other element, and so may determine *the positive side of the surface*. In what follows it will be assumed, unless otherwise stated, that the surface does not cut itself.

269.] Hence we can define also the positive direction of motion for a point passing round the bounding curve of any surface, whether plane or not. For taking an element of the surface part of whose boundary is the elementary arc PP' of the curve, and having chosen the positive side of that element, we determine by our convention the positive direction of the point's motion along PP', and therefore its positive direction of motion round the bounding curve of the surface.

270.] If the bounding curve of a plane area be traced out by a radius vector through a point O not in its plane, the *solid angle* subtended at O by the area may be defined as the s[illegible]rtion

of a spherical surface of unit radius described about O as centre cut out by the radius vector. It may be defined as positive or negative according to the motion of the radius vector, namely positive, if the point of intersection of the radius vector with the bounding curve moves, as seen from O, in the opposite direction to that of the hands of a watch, negative if in the same direction. If the direction of motion of the point be taken as positive or negative with reference to the normal according to the definition in Art. 268, then the solid angle subtended at O by the area is positive or negative according as O is on the positive or negative side of the plane.

The solid angle subtended at O by any finite surface is the sum of the solid angles subtended at O by all the elementary areas into which the surface can be divided. It is, according to this definition, a single-valued function of the position of O.

Stokes's Theorem.

271.] Let $d\sigma$ be an element of a single surface bounded by a closed curve, ds an element of the curve, l, m, n the direction-cosines of the normal to the surface at the point x, y, z, and let P be any function of x, y, and z. Then shall the surface integral $\iint\left(m\frac{dP}{dz}-n\frac{dP}{dy}\right)d\sigma$ taken over the surface be equal to the line integral $\int P\frac{dx}{ds}\,ds$ taken round the curve in the positive direction.

For let P_0 be the value of P at the centre of inertia of the surface element $d\sigma$. Through that centre of inertia let axes be drawn parallel to those of x, y, and z, and let x', y', z' be the co-ordinates referred to these new axes of a point in the curve bounding $d\sigma$. Then the value of P at x', y', z' is

$$P_0 + x'\frac{dP_0}{dx} + y'\frac{dP_0}{dy} + z'\frac{dP_0}{dz},$$

and the line integral $\int P\frac{dx'}{ds'}\,ds'$ taken round this elementary curve becomes

$$P_0\int\frac{dx'}{ds'}\,ds' + \frac{dP_0}{dx}\int x'\frac{dx'}{ds'}\,ds' + \frac{dP_0}{dy}\int y'\frac{dx'}{ds'}\,ds' + \frac{dP_0}{dz}\int z'\frac{dx'}{ds'}\,ds'.$$

The first two terms are severally zero because the elementary curve is closed. The last two terms are equal to $-n\frac{dP_0}{dy}d\sigma$ and $+m\frac{dP_0}{dz}d\sigma$ respectively. Hence

$$\left(m\frac{dP_0}{dz}-n\frac{dP_0}{dy}\right)d\sigma=\int P\frac{dx'}{ds'}ds'.$$

And since we may regard P as constant over $d\sigma$ the theorem is proved for the elementary area $d\sigma$ and its bounding curve.

Hence in the case of a finite surface the surface integral $\iint\left(m\frac{dP}{dz}-n\frac{dP}{dy}\right)d\sigma$ is equal to the sum of all the line integrals $\int P\frac{dx}{ds}ds$ round all the elementary areas into which the surface is divided. But in this summation every part of each line integral is taken twice, once in the positive and once in the negative direction, unless it belong to the final bounding curve; so that all the line integrals cancel each other except those relating to the bounding curve. It follows that, for the whole surface and its bounding curve,

$$\iint\left(m\frac{dP}{dz}-n\frac{dP}{dy}\right)d\sigma=\int P\frac{dx}{ds}ds. \quad . \quad . \quad . \quad . \quad (1)$$

COROLLARY I. The surface integral is zero for any closed surface.

COROLLARY II. If X, Y, Z be any three functions of x, y, and z, by applying the theorem to each of them we may obtain an expression of the form

$$\iint\left\{l\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)+m\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+n\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right\}d\sigma$$
$$=\int\left\{\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)\right\}ds.$$

And if further X, Y, and Z be the components of a vector R, and ϵ be the angle between R and ds, we have

$$\iint\left\{l\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)+m\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+n\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right\}d\sigma$$
$$=\int R\cos\epsilon\, ds.$$

272.] Let r be the vector from the point x, y, z to the origin. Let ρ be a vector drawn from the origin with direction-cosines l, m, n, and making with r the angle θ. Let us so choose the positive side of the plane of r and ρ as that when seen from a point in the normal through the origin on the positive side the shortest way to turn r to coincide with ρ would be in the direction of watch-hand movement. Then the direction-cosines of this normal are

$$\frac{1}{\sin\theta}\frac{ny-mz}{r}, \quad \frac{1}{\sin\theta}\frac{lz-nx}{r}, \quad \frac{1}{\sin\theta}\frac{mx-ly}{r}.$$

In the normal so drawn let us take a vector whose length is proportional to $\frac{\sin\theta}{r^2}$, and let its components be denoted by f, g, h. Then we have

$$\left.\begin{aligned} f &= \frac{ny-mz}{r^3} = \left(m\frac{d}{dz} - n\frac{d}{dy}\right)\frac{1}{r}, \\ g &= \frac{lz-nx}{r^3} = \left(n\frac{d}{dx} - l\frac{d}{dz}\right)\frac{1}{r}, \\ h &= \frac{mx-ly}{r^3} = \left(l\frac{d}{dy} - m\frac{d}{dx}\right)\frac{1}{r}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

If therefore in (1) we make $P = \frac{1}{r}$, and take ρ parallel to the positive normal to the surface at the element $d\sigma$, we obtain the equations

$$\int \frac{1}{r}\frac{dx}{ds}\,ds = \iint \left(m\frac{d}{dz} - n\frac{d}{dy}\right)\frac{1}{r}\,d\sigma = \iint f\,d\sigma.$$

Similarly,
$$\left.\begin{aligned} \iint \frac{1}{r}\frac{dy}{ds}\,ds &= \iint g\,d\sigma, \\ \iint \frac{1}{r}\frac{dz}{ds}\,ds &= \iint h\,d\sigma. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If r be measured to the point ξ, η, ζ instead of to the origin, the foregoing equations become

$$\left.\begin{aligned} f &= \left(n\frac{d}{d\eta} - m\frac{d}{d\zeta}\right)\frac{1}{r}, \\ g &= \left(l\frac{d}{d\zeta} - n\frac{d}{d\xi}\right)\frac{1}{r}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$h=\left(m\frac{d}{d\xi}-l\frac{d}{d\eta}\right)\frac{1}{r},$$

because $\qquad r^2=(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2,$

and therefore $\frac{d}{d\xi}\frac{1}{r}$, $\frac{d}{d\eta}\frac{1}{r}$, and $\frac{d}{d\zeta}\frac{1}{r}$ are the negatives of $\frac{d}{dx}\frac{1}{r}$, $\frac{d}{dy}\frac{1}{r}$, and $\frac{d}{dz}\frac{1}{r}$ respectively.

273.] If with the same meanings as before of θ, f, g, h we make

$$v=\frac{\cos\theta}{r^2},$$

then shall $\qquad -\frac{dv}{dx}=\frac{dh}{dy}-\frac{dg}{dz},\quad -\frac{dv}{d\xi}=\frac{dh}{d\eta}-\frac{dg}{d\zeta};$

with corresponding equations for $\frac{dv}{dy}$, $\frac{dv}{d\eta}$, etc.

For
$$v=\frac{l(\xi-x)+m(\eta-y)+n(\zeta-z)}{r^3}$$
$$=-\left(l\frac{d}{d\xi}+m\frac{d}{d\eta}+n\frac{d}{d\zeta}\right)\frac{1}{r},$$
$$\therefore\quad -\frac{dv}{d\xi}=\left(l\frac{d^2}{d\xi^2}+m\frac{d^2}{d\xi d\eta}+n\frac{d^2}{d\xi d\zeta}\right)\frac{1}{r}$$
$$=\left(m\frac{d^2}{d\xi d\eta}-l\frac{d^2}{d\eta^2}\right)\frac{1}{r}$$
$$-\left(l\frac{d^2}{d\zeta^2}-n\frac{d^2}{d\xi d\zeta}\right)\frac{1}{r},$$

because $\qquad \left(\frac{d^2}{d\xi^2}+\frac{d^2}{d\eta^2}+\frac{d^2}{d\zeta^2}\right)\frac{1}{r}=0$

$$=\frac{dh}{d\eta}-\frac{dg}{d\zeta}.$$

The remaining equations follow by symmetry.

COROLLARY. $\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=\frac{df}{d\xi}+\frac{dg}{d\eta}+\frac{dh}{d\zeta}=0,$

by differentiation of (2) and (4).

274.] If ds be an element of the curve bounding the surface σ, and θ be, as in Art. 272, the angle between the normal to the surface at the point x, y, z, and the vector r from the point x, y, z to ξ, η, ζ, then shall

$$-\frac{d}{d\xi}\iint\frac{\cos\theta}{r^2}d\sigma=\int\frac{(y-\eta)\frac{dz}{ds}-(z-\zeta)\frac{dy}{ds}}{r^3}ds,$$

the surface integral being taken over σ, and the line integral round the bounding curve.

For
$$-\frac{d}{d\xi}\iint\frac{\cos\theta}{r^2}d\sigma = -\iint\frac{d}{d\xi}\frac{\cos\theta}{r^2}d\sigma$$
$$=\iint\left(\frac{dh}{d\eta}-\frac{dg}{d\zeta}\right)d\sigma \text{ (by Art. 273)},$$
remembering that the direction of the normal at x, y, z, is independent of ξ, η, and ζ.

But by Art. 272,
$$\iint h\,d\sigma = \int\frac{1}{r}\frac{dz}{ds}ds,$$
$$\iint g\,d\sigma = \int\frac{1}{r}\frac{dy}{ds}ds.$$

Therefore
$$-\frac{d}{d\xi}\iint\frac{\cos\theta}{r^2}d\sigma = \int\frac{d\frac{1}{r}}{d\eta}\frac{dz}{ds}ds - \int\frac{d\frac{1}{r}}{d\zeta}\frac{dy}{ds}ds$$
$$=\int\frac{(y-\eta)\frac{dz}{ds}-(z-\zeta)\frac{dy}{ds}}{r^3}ds.$$

The expression
$$\frac{(y-\eta)\frac{dz}{ds}-(z-\zeta)\frac{dy}{ds}}{r^3}$$
is the x-component of the line drawn in the positive direction perpendicular to the plane of r and ds, and equal in length to $\frac{\sin\phi}{r^2}$, where ϕ is the angle between r and ds.

275.] We proved in Art. 12 that if u and u' be two functions, *both of lower degree than* $-\frac{1}{2}$, satisfying the conditions $\frac{du}{d\nu}=\frac{du'}{d\nu}$ at all points on a closed surface S, and $\nabla^2 u = \nabla^2 u' = 0$ at all points outside of S, then $u = u'$ at all points on or outside of S. It is now necessary for our purpose to extend this proposition to any two functions of negative degree without restriction.

Let O be any point within S, r the distance from O to any point in space. Then $\frac{u}{r}$ and $\frac{u'}{r}$ are both of lower degree than -1. Therefore by Green's theorem applied to S and infinite external space,

$$\iint u\frac{d}{d\nu}\frac{1}{r}dS-\iiint u\nabla^2\frac{1}{r}\,dy\,dy\,dz$$

$$=\iint\frac{1}{r}\frac{du}{d\nu}dS-\iiint\frac{1}{r}\nabla^2 u\,dx\,dy\,dz.$$

That is,
$$\iint u\frac{d}{d\nu}\frac{1}{r}dS=\iint\frac{1}{r}\frac{du}{d\nu}dS$$

$$=\iint\frac{1}{r}\frac{du'}{d\nu}dS=\iint u'\frac{d}{d\nu}\frac{1}{r}\,dS,$$

or
$$\iint(u-u')\frac{d}{d\nu}\frac{1}{r}\,dS=0.$$

And as this is true for all positions of O within S, it follows necessarily that $u-u'=0$ at all points on S. For if not, $u-u'$ must have a maximum or minimum value at some point on S, suppose P. And by making O approach sufficiently near to P we could make the integral $\iint(u-u')\frac{d}{d\nu}\frac{1}{r}\,dS$ differ from zero.

We have then $u-u'=0$ at all points on S. Also $u-u'=0$ at all points at an infinite distance. Therefore $u-u'$ must be zero at all points in space outside of S. For if not it must have a maximum or minimum value at some point outside of S. And this is impossible by Art. 53, since $\nabla^2(u-u')=0$ at all points outside of S.

CHAPTER XVI.

ON MAGNETIC PHENOMENA*.

ARTICLE 276.] Certain bodies, as, for instance, the iron ore called load-stone, and pieces of steel which have been subjected to certain treatment, are found to possess the following properties, and are called Magnets.

If, near any part of the earth's surface except the Magnetic Poles, a magnet be suspended so as to turn freely about a vertical axis, it will in general tend to set itself in a certain azimuth, and if disturbed from this position it will oscillate about it. An unmagnetised body has no such tendency, but is in equilibrium in all azimuths alike.

It is found that the force which acts on the body tends to cause a certain line in the body, called the Axis of the Magnet, to become parallel to a certain line in space, called the Direction of the Magnetic Force.

The direction of the magnetic force is found to be different at different points of the earth's surface. If the two points in which the axis meets the outer surface of the magnet be called the ends of the magnet, and that end which points in a northerly direction be marked, it is found that the direction in which the axis of the magnet sets itself in general deviates from the true meridian to a considerable extent, and that the marked end points on the whole downwards in the northern hemisphere and upwards in the southern.

The azimuth of the direction of the magnetic force, measured from the true North in the westerly direction, is called the *Variation*, or the *Magnetic Declination*. The angle between the direction of the magnetic force and the horizontal plane is called the *Magnetic Dip*. These two angles determine the direction of

* The introductory portion (Arts. 276–282) of this chapter is taken almost without alteration from Maxwell's 'Electricity and Magnetism,' vol. ii. chap. 1.

the magnetic force, and, when the magnetic intensity is also known, the magnetic force is completely determined. The determination of the values of these three elements at different parts of the earth's surface, the discussion of the manner in which they vary according to the place and time of observation, and the investigation of the causes of the magnetic force and its variations, constitute the science of Terrestrial Magnetism.

277.] Let us now suppose that the axes of several magnets have been determined, and that the end of each which points north has been marked. Then, if one of these be freely suspended, that is in such a way as to be free to turn in all directions about its centre of gravity, the action of its weight being thereby eliminated, and another brought near to it, it is found that the two marked ends repel each other, that a marked and an unmarked end attract each other, and that two unmarked ends repel each other.

If the magnets are in the form of long rods or wires, uniformly and longitudinally magnetised called bar magnets, it is found that the greatest manifestation of force occurs when the end of one magnet is held near the end of the other, and that the phenomena can be accounted for by supposing that like ends of the magnet repel each other, that unlike ends attract each other, and that the intermediate parts of the magnets have no sensible mutual action.

278.] The ends of a long thin magnet such as those just described are commonly called its Poles. In the case of an indefinitely thin magnet, uniformly magnetised throughout its length, the extremities act as centres of force, and the rest of the magnet appears devoid of magnetic action. In all actual magnets the magnetisation deviates from uniformity so that no single points can be taken as the poles. Coulomb, however, by using long thin rods magnetised with care, succeeded in establishing the law of force between two magnetic poles as follows:—

The repulsion between two magnetic poles is in the straight line joining them, and is numerically equal to the product of the strengths of the poles divided by the square of the distance between them.

That is to say, in the case of two ideal bar or needle magnets in the presence of each other the mechanical action between them is exactly the same as if at the poles of each there were placed a charge of electricity, one positive and the other negative, numerically equal to the strength of the pole.

279.] This law, of course, assumes that the strength of each pole is measured in terms of a certain unit, the magnitude of which may be deduced from the terms of the law.

The unit-pole is a pole which points North, and is such that when placed at unit distance from another unit-pole, it repels it with unit of force. A pole which points South is reckoned negative.

If m_1 and m_2 are the strengths of two magnetic poles, if l be the distance between them, and f the force of repulsion, all expressed numerically, then

$$f = \frac{m_1 m_2}{l^2}.$$

Whence it follows that the dimensions of the concrete unit-pole are the same as those of the electrostatic unit of electricity, namely, $\frac{3}{2}$ as regards length, -1 as regards time, and $\frac{1}{2}$ as regards mass. See Chap. XVII *post*.

The accuracy of this law may be considered as having been established by the experiments of Coulomb with the torsion balance, and confirmed by the experiments of Gauss and Weber, and of all observers in magnetic observatories, who are every day making measurements of magnetic quantities, and who obtain results which would be inconsistent with each other if the law of force had been erroneously assumed. It derives additional support from its consistency with the laws of electromagnetic phenomena.

280.] It is not possible to obtain an ideally perfect bar magnet such as we have been considering, and if so obtained it would be equally impossible to maintain its strength unaltered for any length of time, for reasons hereafter to be mentioned. If, however, we imagine such an ideal magnet to exist and its strength to remain always the same, and call this magnet the magnet of reference, then all experimental evidence points to the following conclusions.

(1) That, as has been already implied, if either pole of the magnet of reference were brought near to the middle point of any bar magnet no mechanical action would be apparent, and such action would be feeble at all points near to the middle of the magnet.

(2) If the bar magnet under investigation were broken into two or more pieces of any lengths equal or unequal, then each of the pieces thus obtained would form a short magnet whose positive and negative poles are at those respective extremities of each short magnet nearer in the unbroken state to the corresponding poles of the original magnet.

(3) It is impossible by any process whatever to obtain a magnet whose poles are of unequal strength, and therefore impossible to isolate a pole.

The multiplication of magnets by fracture and creation of mechanical energy is not inconsistent with the conservation of energy, because after fracture and before separation the adjacent poles of the several magnets neutralise each other, and the act of separation involves mechanical work.

Magnetic Theory.

281.] The resemblance mentioned above (Art. 278) between the mutual action of bar magnets and of bodies charged with equal and opposite quantities of electricity at the poles of these magnets could hardly fail to suggest the conception of magnetic matter or magnetic fluids endowed with properties of mutual action according to exactly the same laws as the supposed electric fluids; and indeed such an hypothesis has proved capable of explaining some of the phenomena of magnetism as successfully as the two-fluid hypothesis explains the phenomena of statical electricity. At the same time such fluids have even less claim to be regarded as physical realities in the magnetic theory than they have in the electrical. They are nothing more than mathematical fictions of great use in the enunciation and systematisation of the laws of magnetic phenomena.

282.] The two-fluid theory of magnetism assumes the existence of two magnetic fluids called positive and negative respectively,

and attracting or repelling according to exactly the same laws as govern the actions of the positive and negative electric fluids.

In the magnetic theory, however, there is nothing that corresponds to a body charged with electricity. The molecules of all substances which are capable of manifesting magnetic action are supposed to be charged with exactly equal quantities of both fluids, and it is to the separation of these opposite fluids within each molecule that the phenomena of magnetisation are ascribed. Each separate molecule is thus regarded, when magnetised, as having acquired the property of polarisation, that is to say there is a certain line moving with the molecule, such that if by turning the molecule the direction of this line is reversed, then the magnetic action between this molecule and the surrounding field is exactly reversed also. The particular mode of separation of the fluids within each molecule does not enter into consideration, any more than the particular shape of the molecule. As a very simple case we might suppose the molecules of a substance to be equal and similar prisms or cylinders, and the separation in each to take place by the aggregation of all the positive fluid at one end, and all the negative fluid at the other, i. e. by equal positive and negative superficial distributions at opposite ends. Each molecule would thus become an elementary bar magnet as above defined, the end on which the positive distribution was situated being the positive pole. If a finite prism were built up of a very great number of these molecules placed end to end, the positive pole of any one of them being in contact with the negative pole of the succeeding one, the mechanical action of equal and opposite contiguous poles of contiguous molecules would neutralise each other, and there would remain a bar magnet of finite length, the strength of whose poles was exactly the same as the strength of the poles of the molecular magnets. This conception however of the particular shape of the constituent molecules or of the particular mode of fluid distribution within each one of them, is not, as above said, essential to the theory.

283.] In Chaps. X and XI, Vol. I, we discussed the properties of a medium consisting of an infinite number of discrete infinitesimal molecules of any shape whatever, each of them containing

either in solid or superficial distribution exactly equal quantities of the positive and negative electric fluids, and we proved that if $\phi\, dx\, dy\, dz$ were the algebraic sum of the mass of the fluids within the elementary volume $dx\, dy\, dz$ in the neighbourhood of the point x, y, z in such a medium, and if σ_x were equal to the triple integral $\iiint x\,\phi\, dx\, dy\, dz$ taken over an unit volume throughout which the distribution of molecules and of the electricity within each molecule is uniform, and the same as it is in the neighbourhood of the point x, y, z in the actual medium, with similar meanings for σ_y and σ_z, then $\sigma_x, \sigma_y, \sigma_z$ are components of a certain vector σ; and that if a plane were drawn through x, y, z the direction-cosines of whose normal were l, m, n, then the algebraical mass of the fluids within or upon the molecules intersected by this plane, and situated on the positive side of this plane, is $l\sigma_x + m\sigma_y + n\sigma_z$.

The vector σ so obtained we defined as the *polarisation* of the medium at the point x, y, z, and the quantities σ_x, σ_y, and σ_z as the components of polarisation at that point.

We proved also that if $\rho\, dx dy dz$ were the algebraical sum of the electrical fluids within the volume element $dx dy dz$ in such a medium, i. e. that if ρ were the electric volume density at the point x, y, z, then

$$\rho = -\left\{\frac{d\sigma_x}{dx} + \frac{d\sigma_y}{dz} + \frac{d\sigma_z}{dz}\right\}.$$

And that if over any plane in such a medium whose normal direction-cosines were l, m, n there were superficial electric distribution, then $\sigma_x, \sigma_y, \sigma_z$, and σ must be discontinuous at points on the plane; and that if σ_x and σ'_x, σ_y and σ'_y, σ_z and σ'_z were the values of these quantities at any point of the plane on opposite sides of it, then the superficial density of electrical distribution over the plane at that point would be

$$l\,(\sigma_x - \sigma'_x) + m\,(\sigma_y - \sigma'_y) + n\,(\sigma_z - \sigma'_z).$$

284.] If in the medium above described the electric fluids be replaced by magnetic fluids, we arrive at the conception of a magnetised mass in the theory we are now developing. In

conformity with the usual notation we shall replace the symbols σ_x, σ_y, and σ_z by A, B, and C respectively, and the symbol σ by I. Instead also of the terms polarisation and components of polarisation as denoted by the aforesaid symbols, we shall employ the terms magnetisation and components of magnetisation respectively. It will be understood that we are here treating of the effects instantaneously produced by a system of polarised or magnetised molecules, and not of the means by which their polarisation may be produced, maintained, or destroyed.

285.] If we assume the existence of these polarised or magnetised molecules, it follows that there will be a magnetic potential and magnetic force at every point in the field of a magnetised mass, each in all respects possessing the properties investigated in Chap. III, Vol. I, and that with the molecular arrangement and distribution just described, if V be the potential at any point in the field,

$$V = -\iiint \frac{1}{r}\Big(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\Big)\,dx\,dy\,dz + \iint \frac{lA + mB + nC}{r}\,dS, \quad (1)$$

where r is the distance of the element $dx\,dy\,dz$ of the mass, or of its surface element dS from the point ξ, η, ζ at which the potential is estimated, and the volume and superficial integrations extend throughout the volume of S and over its surface respectively. The surface-integral may be more accurately written in the form

$$\Sigma \iint \frac{l(A - A') + m(B - B') + n(C - C')}{r}\,dS, \quad . \quad . \quad . \quad (2)$$

the summation Σ extending over all the elementary surfaces at which there is discontinuity in the values of A, B, and C, and the quantities l, m, n being direction-cosines of the normal of such element in each case; the simpler form first written contemplating the case of a single magnet with continuous magnetisation within its volume and surrounded by a non-magnetised medium.

286.] Restricting ourselves to case (1), and integrating each term for x, y, and z respectively, we obtain the equation

$$V = \iiint \Big\{A\frac{d}{dx} + B\frac{d}{dy} + C\frac{d}{dz}\Big\}\frac{1}{r}\,dx\,dy\,dz.$$

magnetisation. If the magnetisation be uniform throughout in intensity as well as direction, the equation becomes

$$V = -I\frac{d}{di}\iiint\frac{dx\,dy\,dz}{r} = -I\frac{dV_0}{di}, \quad . \quad . \quad . \quad . \quad (2)$$

where V_0 is the potential at ξ, η, ζ of a mass of uniform density unity occupying the volume of the given substance.

290.] If therefore we know the potential of any given electrical distribution at any point, either uniform or varying according to any law, we can at once by mere differentiation determine the potential at that point of a corresponding magnetic distribution of given intensity and uniform direction.

For example, the potential of a sphere (rad. a) of density unity, at the point ξ, η, ζ distant r from the centre as origin, is

$$\frac{4\pi a^3}{3r} \text{ if } P \text{ be external to the sphere,}$$

and $$2\pi\left(a^2 - \frac{r^2}{3}\right) \text{ if } P \text{ be within the sphere.}$$

Therefore the potential of a sphere of uniform magnetisation I parallel to the axis of x is

$$I\frac{4\pi}{3}\cdot\frac{a^3}{r^3}\cdot\xi \text{ for an external point,}$$

and $$I\frac{4\pi}{3}\xi \text{ for an internal point.}$$

And the magnetic force has for its components in the former case

$$-\frac{4\pi}{3}Ia^3\left\{\frac{1}{r^3} - \frac{3\xi^2}{r^5}\right\}; \quad +\frac{4\pi a^3\xi\eta}{r^5}, \quad +\frac{4\pi a^3\xi\zeta}{r^5},$$

and in the latter case $\quad -\frac{4\pi}{3}I, 0, 0$
respectively.

291.] Again, the potential V_0 of an ellipsoid of uniform density unity at any internal point ξ, η, ζ referred to the principal axes as axes of co-ordinates is known to be given by the equation

$$V_0 = C - \tfrac{1}{2}(L\xi^2 + M\eta^2 + N\zeta^2),$$

where L, M, N are certain known functions of the semi-axes, a, b, and c. Therefore the potential of an ellipsoidal mass with uniform

magnetisation I parallel to the line i whose direction-cosines are l, m, n at the internal point ξ, η, ζ is

$$-I\frac{dV_0}{di} \quad \text{or} \quad I(Ll\xi+Mm\eta+Nn\zeta)=(LA\xi+MB\eta+NC\zeta),$$

where A, B, and C are the components of magnetisation at each point of the mass.

292.] We now proceed to consider certain particular cases of magnetisation, one of which, namely that of a uniform bar magnet, has already been noticed.

An Elementary Magnet.

We have seen that the potential at the point ξ, η, ζ due to a magnetic mass may be expressed in the form $\iiint I\frac{\cos\theta}{r^2}\,dx\,dy\,dz$, where I is the resultant magnetisation, r the vector from x, y, z to ξ, η, ζ, and θ the angle between I and r. If the dimensions of the magnet be infinitesimal, this may be put in the form $ISh\frac{\cos\theta}{r^2}$, when S is the transverse section, h the length of the magnet in direction of I.

The quantity IS is called the *strength of the pole*, and ISh *the moment* of the magnet.

Bar magnets of uniform magnetisation may be regarded as elementary magnets, so far as relates to points in the field whose distance from them is great compared with their linear dimensions.

293.] *Definitions.* A line either straight or curved drawn through any magnetised mass, so that its tangent coincides at every point with the direction of the resultant magnetisation at the point, is called a *line of magnetisation.*

A tubular surface constructed in any magnetised mass, so that the line of magnetisation at every point on the surface lies on the surface, is called *a tube of magnetisation.* And when the transverse section of the tube is indefinitely small, it is called an *elementary tube of magnetisation.*

If throughout the space bounded by any closed surface S within a magnetised mass the distribution be such that

$$\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}=0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(A)}$$

at every point, and therefore the integral

$$\iiint\left\{\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right\}dx\,dy\,dz$$

taken throughout the space is zero, it follows that

$$\iint\{lA+mB+nC\}\,dS$$

taken over S is zero, l, m, n being the direction-cosines of the normal to S.

If S be formed of a tube of magnetisation and two transverse sections S_1 and S_2, then since $lA+mB+nC=0$ at each point on S except points on S_1 or S_2, it follows that if I_1 be the resultant magnetisation on S_1, and I_2 on S_2, measured in both cases outwards from S,

$$\iint I_1\,dS_1+\iint I_2\,dS_2=0.$$

A magnetic distribution satisfying the condition (A) is called *solenoidal*.

A Magnetic Solenoid.

294.] A magnetised mass in which the distribution is solenoidal bounded by an elementary tube of magnetisation, is called a *simple magnetic solenoid*. Since V, the potential at the point ξ, η, ζ of any magnetised mass, is given by the equation

$$V=\iiint\frac{I\cos\theta}{r^2}\,dx\,dy\,dz,$$

if we replace the volume element $dx\,dy\,dz$ by $S\,dh$, where S is the transverse section of the tube perpendicular to its axis at the point whose distance measured along the axis of the tube from a fixed point in the axis is h, we have

$$V=\int\frac{I\cos\theta}{r^2}\,Sdh=\int\frac{IS}{r^2}\frac{dr}{dh}\,dh.$$

The product IS at any point of the solenoid is called the strength

...he potential V at P due to any magnetised
... given by the equation

$$\iiint I \frac{\cos\theta}{r^2}\, dx\, dy\, dz,$$

...magnetisation at the point x, y, z of the
...e from x, y, z to P, and θ the angle between
... If we replace the volume-element
... $h\,dS$, where dS is an element of one of
... of the shell, and h the thickness of the

$$V = \iint \frac{Ih\cos\theta}{r^2}\, dS$$

$$\ldots \iint \frac{\cos\theta}{r^2}\, dS,$$

...ength of the shell. Evidently from the
...n to $\cos\theta$, this integral will be positive
... face of the shell presented to O
... magnetisation.

... the *solid angle* subtended by the

...shell on which is positive magneti-
... magnetisation is measured, is the
... 268).

...ote by ω the solid angle subtended by
...question, $V = \phi\omega$; where the sign is
...ntioned. As P moves from a point
... round the boundary of the shell to a
... the opposite face, V passes from $-2\pi\phi$
... to $-2\pi\phi$ according as the passage is
...positive or from the positive to the
...and if the passage be through the
... decrease by $4\pi\phi$ takes place in the
...point on one face to a very near
... this result does not imply dis-
... inasmuch as the thickness of the
...ite.

296.] If the distribution of magnetisation in any magnetic mass be such that all the lines of magnetisation can be cut orthogonally by a system of surfaces, we know that the components of magnetisation, A, B, C, must satisfy the equation of condition,

$$A\left\{\frac{dB}{dz}-\frac{dC}{dy}\right\}+B\left(\frac{dC}{dx}-\frac{dA}{dz}\right)+C\left\{\frac{dA}{dy}-\frac{dB}{dx}\right\}=0$$

at every point. When this condition is satisfied, the mass between any two of the surfaces may be divided into elementary portions, each bounded by a tube of magnetisation, and two surfaces which cut the tube and all the lines of magnetisation within the tube at right angles, the distance between the transverse surfaces measured along a normal to either of them at every point being indefinitely small. A magnetic mass bounded by two surfaces satisfying this condition is called a *magnetic shell*, the normal distance between the surfaces at any point is called the *thickness* of the shell at the point, and the product of this thickness into the resultant magnetisation at the point is called the *strength* of the shell at that point. When the strength of the shell is uniform throughout, it is called a *simple magnetic shell*, otherwise a *complex magnetic shell*.

297.] A simple magnetic shell may therefore be otherwise defined as a thin shell of magnetised matter in which the magnetisation is everywhere normal to the surface, and its intensity at any point multiplied by the thickness of the shell at that point is uniform throughout. The product thus found is the strength of the shell. If it be denoted by ϕ, and if l, m, n be the direction-cosines of the normal to the shell, h its thickness, evidently $Ah=l\phi$, $Bh=m\phi$, $Ch=n\phi$, $Ih=\phi$. If, the arrangement being in other respects the same, the above-mentioned product is not uniform throughout, that is if the strength varies from point to point, it constitutes a complex magnetic shell. A complex magnetic shell may be conceived as made up of simple magnetic shells superposed and overlapping one another, in the same way as a complex solenoid may be conceived as composed of overlapping simple solenoids.

298.] To find the potential at any point P (ξ, η, ζ) of a simple

magnetic shell. The potential V at P due to any magnetised mass is, as above shown, given by the equation

$$V = \iiint I \frac{\cos\theta}{r^2}\, dx\, dy\, dz,$$

where I is the resultant magnetisation at the point x, y, z of the mass, r the distance from x, y, z to P, and θ the angle between the direction of I and r. If we replace the volume-element $dx\, dy\, dz$ by its equivalent hdS, where dS is an element of one of the transverse surfaces of the shell, and h the thickness of the shell at dS, we have

$$V = \iint \frac{Ih\cos\theta}{r^2}\, dS$$

$$= \phi \iint \frac{\cos\theta}{r^2}\, dS,$$

where ϕ denotes the strength of the shell. Evidently from the interpretation above given to $\cos\theta$, this integral will be positive or negative according as the face of the shell presented to O be that of positive or negative magnetisation.

The integral $\iint \frac{\cos\theta}{r^2}\, dS$ is the *solid angle* subtended by the shell at O. The side of the shell on which is positive magnetisation, or towards which the magnetisation is measured, is the *positive side* of the surface (Art. 268).

299.] If therefore we denote by ω the solid angle subtended by the shell at the point in question, $V = \phi\omega$; where the sign is determined as just now mentioned. As P moves from a point close to the shell on one face round the boundary of the shell to a point close to the shell on the opposite face, V passes from $-2\pi\phi$ to $+2\pi\phi$ or from $+2\pi\phi$ to $-2\pi\phi$ according as the passage is from the negative to the positive or from the positive to the negative face of the shell; and if the passage be through the shell the same increase or decrease by $4\pi\phi$ takes place in the value of V on passing from a point on one face to a very near point on the opposite face, but this result does not imply discontinuity in the value of V, inasmuch as the thickness of the shell, although small, is finite.

300.] The potential of a magnetic shell at any point not in its substance being $\phi \iint \frac{\cos\theta}{r^2} d\sigma$, it follows that the x-component of force at the point ξ, η, ζ is

$$a = -\phi \frac{d}{d\xi} \iint \frac{\cos\theta}{r^2} d\sigma$$

$$= \phi \int \frac{dz}{ds} \frac{d}{d\eta} \frac{1}{r} ds - \phi \int \frac{dy}{ds} \frac{d}{d\zeta} \frac{1}{r} ds$$

$$= \phi \int \left(\frac{dz}{ds} \frac{y-\eta}{r^3} - \frac{dy}{ds} \frac{z-\zeta}{r^3} \right) ds$$

by Art. 274, with corresponding expressions for β and γ.

301.] Let us write

$$\phi \int \frac{1}{r} \frac{dx}{ds} ds = F,$$

$$\phi \int \frac{1}{r} \frac{dy}{ds} ds = G,$$

$$\phi \int \frac{1}{r} \frac{dz}{ds} ds = H.$$

Then also
$$F = \phi \iint \left(m \frac{d}{dz} - n \frac{d}{dy} \right) \frac{1}{r} dS$$

$$= \iint \left(B \frac{d}{dz} - C \frac{d}{dy} \right) \frac{1}{r} dS.$$

Similarly
$$G = \iint \left\{ C \frac{d}{dx} - A \frac{d}{dz} \right\} \frac{1}{r} dS,$$

$$H = \iint \left\{ A \frac{d}{dy} - B \frac{d}{dx} \right\} \frac{1}{r} dS,$$

and
$$a = \frac{dH}{d\eta} - \frac{dG}{d\zeta},$$

$$\beta = \frac{dF}{d\zeta} - \frac{dH}{d\xi},$$

$$\gamma = \frac{dG}{d\xi} - \frac{dF}{d\eta}.$$

302.] Again, let the point ξ, η, ζ be on another surface S' not cutting the shell, bounded by a closed curve; let l', m', n' be the direction-cosines of the normal to S'.

Then
$$\iint \{l'a+m'\beta+n'\gamma\}\, d\sigma'$$
$$=\iint\left\{m'\frac{dF}{d\zeta}-n'\frac{dF}{d\eta}\right\} dS'$$
$$+\iint\left\{n'\frac{dG}{d\xi}-l'\frac{dG}{d\zeta}\right\} dS'$$
$$+\iint\left\{l'\frac{dH}{d\eta}-m'\frac{dH}{d\xi}\right\} dS',$$

or
$$\iint\{l'a+m'\beta+n'\gamma\}\,dS'=\int\left\{F\frac{dx}{ds'}+G\frac{dy}{ds'}+H\frac{dz}{ds'}\right\} dS',$$

the integrals being taken round the bounding curve of the surface S'.

303.] If the distribution of magnetisation throughout any mass be such that it may be divided into simple magnetic shells each of which is either closed or terminates in the surface of the magnet, the magnetisation is said to be *lamellar*.

In lamellar magnetisation there exists a function ϕ of the coordinates such that

$$A=\frac{d\phi}{dx}, \quad B=\frac{d\phi}{dy}, \quad C=\frac{d\phi}{dz}.$$

Fig. 44.

For if a curve of any form be drawn through such a mass, and if it cut two consecutive shell faces, S and S', in the points P and P' as in the figure, and if the coordinates of these points be x, y, z and $x+dx$, $y+dy$, $z+dz$, and if di denote the thickness of the shell which is parallel to the resultant magnetisation, we have

$$di=\frac{A}{I}dx+\frac{B}{I}dy+\frac{C}{I}dz,$$

or
$$A\,dx+B\,dy+C\,dz=I\,di.$$

Now $I\,di$ is the increase of the sum of the strengths of all the shells traversed by the curve from a fixed point in it, due to the

element PP' of the curve, such increase being reckoned positive when we proceed along the curve in the direction of magnetisation. If this sum be called ϕ, it is clear that with the supposed constitution of the magnet ϕ is independent of the form of the curve from the fixed point to P, and is a function of x, y, z only. Therefore

$$A\,dx + B\,dy + C\,dz = d\phi,$$

or

$$A = \frac{d\phi}{dx}, \quad B = \frac{d\phi}{dy}, \quad C = \frac{d\phi}{dz}.$$

The function ϕ is called the *potential of magnetisation*. It must be carefully distinguished from the magnetic potential.

304.] To find the potential at any point of any lamellarly magnetised substance.

If ξ, η, ζ be the coordinates of the point and A, B, C the components of magnetisation at the point x, y, z in the substance, and V the required potential, we know that

$$V = \iiint \left\{ A\frac{d}{dx} + B\frac{d}{dy} + C\frac{d}{dz} \right\} \frac{dx\,dy\,dz}{r},$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$, and the integration is taken throughout the mass.

In this case

$$A = \frac{d\phi}{dx}, \quad B = \frac{d\phi}{dy}, \quad C = \frac{d\phi}{dz},$$

where ϕ is the potential of magnetisation.

Therefore by Green's theorem

$$V = \iint \phi \frac{d}{dv} \cdot \frac{1}{r}\, dS - \iiint \phi \nabla^2 \frac{1}{r}\, dx\,dy\,dz,$$

the symbols having their ordinary signification.

If θ be the angle between the normal to dS measured outwards and the line r drawn from dS to ξ, η, ζ, the equation becomes

$$V = \iint \frac{\phi \cos\theta}{r^2} \cdot dS - \iiint \phi \nabla^2 \frac{1}{r}\, dx\,dy\,dz.$$

If ξ, η, ζ be without the mass, $\nabla^2 \frac{1}{r}$ is everywhere zero, and the equation becomes

$$V = \iint \frac{\phi \cos\theta}{r^2} \cdot dS.$$

If the point ξ, η, ζ be within the mass, then the equation for V becomes

$$V = \iint \frac{\phi \cos\theta}{r^2} dS + 4\pi(\phi),$$

where (ϕ) is the value of ϕ at ξ, η, ζ.

The double integral $\iint \frac{\phi\cos\theta}{r^2} dS$ is generally represented by Ω. The values of Ω for two points close to the surface S, one just within and the other just without the mass, clearly differ by $4\pi(\phi)$, where (ϕ) is the value of ϕ within the mass close to the point; whence it follows that the value of V is continuous on crossing the surface, as it should be by Chap. III, inasmuch as it is the potential of matter of finite density.

The Energy of a Magnetic System.

305.] It is proved in Vol. I, Art. 166, that the potential energy of an electric system is given by the integral $\frac{1}{2}\iiint V\rho\, dx\, dy\, dz$, where V is the potential of the system, and ρ the volume density of electricity, at the point x, y, z, the triple integral being replaced by the double integral $\frac{1}{2}\iint V\sigma\, dS$ for surfaces of superficial electrification. By reasoning in all respects similar to that used in obtaining the above-mentioned result, we obtain for the potential energy of any magnetic system, so far as concerns the magnetic forces alone, the expression

$$-\tfrac{1}{2}\iiint\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right) V\, dx\, dy\, dz$$
$$+\tfrac{1}{2}\iint (lA + mB + nC)\, V dS,$$

the volume integral extending over the substance of the magnetised bodies and the surface integral over their bounding surfaces. This is the work done in constructing the system against the magnetic forces. In an actual magnetised body it may be the case that other intermolecular forces are called into play in constructing the magnet. The above expression does not include work done against such forces.

Again, the *relative* potential energy of one portion of an

electric field with reference to the other portion is $\iiint V\rho\,dx\,dy\,dz$, where ρ is the volume density of the first, and V the potential of the second portion at the point x, y, z. Similarly the relative potential energy of one system of magnetised matter in the field of another system is

$$-\iiint\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)V\,dx\,dy\,dz+\iint(lA+mB+nC)\,V\,dS,$$

in which A, B, and C relate to the first and V to the second system.

This expression is equivalent to

$$\iiint\left(A\frac{dV}{dx}+B\frac{dV}{dy}+C\frac{dV}{dz}\right)dx\,dy\,dz.$$

It is this relative potential with which we shall be mainly concerned in the following investigations.

306.] On the potential energy of a magnetised mass in a field of uniform force.

If X, Y, Z be the components of the force, W the energy required, it follows that

$$W=-X\iiint A\,dx\,dy\,dz-Y\iiint B\,dx\,dy\,dz-Z\iiint C\,dx\,dy\,dz.$$

If we denote the integrals $\iiint A\,dx\,dy\,dz$, $\iiint B\,dx\,dy\,dz$, and $\iiint C\,dx\,dy\,dz$ by lK, mK, and nK respectively, the above expression becomes

$$W=-K\iiint(lX+mY+nZ)\,dx\,dy\,dz.$$

If, further, $l^2+m^2+n^2=1$, the quantity K is called the *magnetic moment* of the magnet, and the line whose direction-cosines are l, m, n is called the *axis of the magnet*.

If R denote the constant force, and ϵ the angle between F and the axis of the magnet, the potential energy W is given by

$$W=-RK\cos\epsilon.$$

Any region on the earth's surface is sensibly a field of uniform magnetic force. If ϕ and θ be the azimuth and horizontal

inclination of the axis of the magnet, and δ and ζ the corresponding quantities for the magnetic force, then the axis of z being vertical and that of x in the meridian,

$$X = R\cos\zeta\cos\delta, \quad Y = R\cos\zeta\sin\delta, \quad Z = R\sin\zeta,$$
$$l = \cos\theta\cos\phi, \quad m = \cos\theta\sin\phi, \quad n = \sin\theta,$$

and therefore $\quad W = -KR\{\cos\zeta\cos\theta\cos(\phi-\delta)+\sin\zeta\sin\theta\}.$

If therefore the magnet be suspended by its centre of inertia, so as to be free to turn about that point, the generalised component of force tending to increase ϕ, or the moment of the force tending to turn the magnet round a vertical axis, is

$$-\frac{dW}{d\phi}, \quad \text{or} \quad -KR\cos\zeta\cos\theta\sin(\phi-\delta);$$

and similarly the moment of the force tending to increase the inclination of the axis to the horizontal plane is

$$-\frac{dW}{d\theta}, \quad \text{or} \quad KR\{\sin\zeta\cos\theta-\cos\zeta\sin\theta\cos(\phi-\delta)\}.$$

307.] To find the magnetic potential energy of any lamellarly magnetised substance in a magnetic field.

If W be the potential energy required,

$$\begin{aligned} W &= \iiint\left(A\frac{dV}{dx}+B\frac{dV}{dy}+C\frac{dV}{dz}\right)dx\,dy\,dz \\ &= \iiint\left(\frac{d\phi}{dx}\frac{dV}{dx}+\frac{d\phi}{dy}\frac{dV}{dy}+\frac{d\phi}{dz}\frac{dV}{dz}\right)dx\,dy\,dz \\ &= \iint\phi\frac{dV}{d\nu}\,dS-\iiint\phi\nabla^2 V\,dx\,dy\,dz, \end{aligned}$$

ϕ being the potential of magnetisation at the point x, y, z of the mass, and V the potential of the field at that point.

If the potential energy required be relative to a field of magnetisation entirely without the mass, then $\nabla^2 V = 0$, and the equation becomes

$$W = \iint\phi\frac{dV}{d\nu}dS,$$

$$\text{or} \quad W = -\iint\phi(l\alpha+m\beta+n\gamma)\,dS,$$

where α, β, γ are the components of force due to the field at the

element dS, and l, m, n are the direction-cosines of the normal to the element.

If the mass be bounded by a tube of magnetisation and the transverse surfaces S_1 and S_2, each of them everywhere at right angles, to lines of magnetisation,

$$W = \phi_1 \iint (l_1 a_1 + m_1 \beta_1 + n_1 \gamma_1)\, dS_1 - \phi_2 \iint (l_2 a_2 + m_2 \beta_2 + n_2 \gamma_2)\, dS_2,$$

the surface integrals being taken over S_1 and S_2 respectively and the normals being measured in both cases from S_1 to S_2.

If the surfaces S_1 and S_2 be very near to each other so that the mass constitutes a uniform magnetic shell of normal thickness i,

$$\phi_2 - \phi_1 = \frac{d\phi}{d\nu} \cdot i = Ii = \Phi,$$

where Φ is the strength of the shell, and

$$W = -\,\Phi \iint (la + m\beta + n\gamma)\, dS.$$

If the field be that of another uniform magnetic shell of strength Φ', we know from Art. 301 that

$$\iint (la + m\beta + n\gamma)\, d\sigma = \int \left(F' \frac{dx}{ds} + G' \frac{dy}{ds} + H' \frac{dz}{ds}\right) ds,$$

where

$$F' = \Phi' \int \frac{1}{r} \frac{dx'}{ds'}\, ds', \quad G' = \Phi' \int \frac{1}{r} \frac{dz'}{ds'}\, ds', \quad H' = \Phi' \int \frac{1}{r} \frac{dz'}{ds'}\, ds',$$

the integrals being taken round the contour of the shell Φ', therefore

$$W = -\,\Phi\Phi' \iint \frac{1}{r} \left(\frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'}\right) ds\, ds'$$

$$= -\,\Phi\Phi' \iint \frac{\cos\epsilon}{r}\, ds\, ds',$$

where ϵ is the angle between the elements ds and ds'.

If the energy required be that of the lamellarly magnetised mass in its own field, then we have by Art. 304,

$$W = \frac{1}{2} \iiint \left(\frac{d\phi}{dx} \cdot \frac{d\Omega}{dx} + \frac{d\phi}{dy}\frac{d\Omega}{dy} + \frac{d\phi}{dz}\frac{d\Omega}{dz}\right)$$

$$+ 2\pi \iiint \left(\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right) dx\, dy\, dz$$

$$= \tfrac{1}{2}\iint \phi \frac{d\Omega}{d\nu} dS + 2\pi \iiint I^2 dx dy dz$$
$$= -\tfrac{1}{2}\iint \phi (l\alpha + m\beta + n\gamma)\, dS + 2\pi \iiint I^2 dx dy dz.$$

And in the case of a uniform shell of strength Φ this becomes as above

$$-\tfrac{1}{2}\Phi^2 \iint \frac{\cos\epsilon}{r}\, ds\, ds' + 2\pi\Phi \iint I dS,$$

the former of the two double integrals being taken for each pair of elements of the contour of the shell.

These two terms are of the same order of magnitude. The energy therefore is not in this case represented by the integral

$$-\tfrac{1}{2}\phi^2 \iint \frac{\cos\epsilon}{r}\, ds ds', \quad \text{or} \quad \tfrac{1}{2}\iint \phi \frac{d\Omega}{d\nu}\, dS.$$

The whole energy of the mass placed in the given external field is

$$\iint \phi \left(\frac{dV}{d\nu} + \tfrac{1}{2}\frac{d\Omega}{d\nu}\right) dS + 2\pi \iiint I^2 dx dy dz.$$

308.] On the potential energy of a given magnetised mass in the field of an elementary magnet.

If x, y, z be the middle point of the axis of the elementary magnet, M its moment, θ the angle between its axis and the line r drawn from x, y, z to the point ξ, η, ζ in the magnetic mass, we know from Art. 298 above that the potential V of the elementary magnet at ξ, η, ζ is $\dfrac{M\cos\theta}{r^2}$, or

$$M\frac{\{l(\xi - x) + m(\eta - y) + n(\zeta - z)\}}{r^3},$$

l, m, n, being direction-cosines of the axis of the elementary magnet, and M its moment.

If this axis be denoted by the symbol h_1, the last expression is $M\dfrac{d}{dh_1}\left(\dfrac{1}{r}\right)$ in the notation of Vol. I, chap. II; therefore

$$V = M\frac{d}{dh_1}\left(\frac{1}{r}\right).$$

If therefore W be the potential energy of the whole magnetised mass in the field of the elementary magnet,

$$W = M\iiint \left\{A\frac{d}{d\xi} + B\frac{d}{d\eta} + C\frac{d}{d\zeta}\right\}\frac{d}{dh_1}\left(\frac{1}{r}\right) d\xi\, d\eta\, d\zeta,$$

A, B, and C being the components of magnetisation of the mass at ξ, η, ζ and the integration being taken throughout the mass.

If the magnetised mass be also an elementary magnet we may regard it as consisting of the single element $d\xi\, d\eta\, d\zeta$, which may also be written as ka, where k is the length of the secondary elementary magnet and a its transverse section, so that W is determined by the equation

$$W = M\left(A\frac{d}{d\xi} + B\frac{d}{d\eta} + C\frac{d}{d\zeta}\right)\frac{d}{dh_1}\left(\frac{1}{r}\right)ka,$$

or $$W = M\left(\lambda\frac{d}{d\xi} + \mu\frac{d}{d\eta} + \nu\frac{d}{d\zeta}\right)\cdot\frac{d}{dh_1}\left(\frac{1}{r}\right)\cdot Ika,$$

where λ, μ, ν are the direction-cosines of the axis, and I the intensity of magnetisation, of the second magnet.

Now $Ika = M_2$ the moment of the second magnet, and

$$\left(\lambda\frac{d}{d\xi} + \mu\frac{d}{d\eta} + \nu\frac{d}{d\zeta}\right) = \frac{d}{dh_2}$$

if h_2 be a line through ξ, η, ζ coincident with the second axis.

Therefore $$\begin{aligned} W &= M_1 M_2 \frac{d}{dh_2}\cdot\frac{d}{dh_1}\left(\frac{1}{r}\right) \\ &= M_1 M_2 \frac{d^2}{dh_2\, dh_1}\left(\frac{1}{r}\right). \end{aligned}$$

If $\mu_{1,2}$ be the cosine of the angle between the two axes and λ_1, λ_2 be the cosines of the angles they make respectively with r, we get

$$W = \frac{M_1 M_2}{r^3}\cdot(\mu_{1\,2} - 3\lambda_1\lambda_2).$$

And from this equation we may determine the force on either magnet in any direction and the couples round any given axis arising from their mutual action by the ordinary methods of generalised coordinates.

CHAPTER XVII.

MAGNETIC INDUCTION AND INDUCED MAGNETISM.

ARTICLE 309.] In Art. 191, we proved that when an infinite plane is situated in a uniform medium of polarised molecules, whose polarisation normal to the plane is σ, the average force at any point on the plane arising from the molecules intersected by the plane is $-4\pi\sigma$.

If, therefore, a plane element dS be drawn through any point P of a magnetic mass, large in comparison with the superficial dimensions of a magnetised molecule, but so small that the polarisation of the mass is sensibly the same as that at P all over dS, the force at P normal to dS arising from the magnetised molecules intersected by dS will be $-4\pi\sigma$, where σ is the magnetic polarisation normal to dS at P, or in the notation now adopted the force is

$$-4\pi(lA+mB+nC),$$

and the flux of force over dS is

$$-4\pi(lA+mB+nC)dS.$$

Now, if α, β, γ be the components of the total magnetic force at P, the total flux of force over dS is $(l\alpha+m\beta+n\gamma)dS$.

Hence it follows that the flux of force over dS arising from the magnetism in the field when the molecules intersected by dS are removed is

$$l(\alpha+4\pi A)+m(\beta+4\pi B)+n(\gamma+4\pi C)\,dS.$$

The force whose flux is thus determined is called the *magnetic induction at P* normal to dS. Its components parallel to the axes of reference are

$$\alpha+4\pi A,\quad \beta+4\pi B,\quad \gamma+4\pi C.$$

They are generally denoted by the symbols a, b, and c re-

spectively, and are called the ***components of magnetic induction at P.***

310.] If any closed surface S be drawn in space, the total flux of the magnetic induction through S is zero. For the quantity of magnetic matter within S is

$$-\iint (lA+mB+nC)\,dS.$$

Therefore, by Art. 42, the total flux of the magnetic force through S, or

$$\iint \{l\alpha+m\beta+n\gamma\}\,dS,$$

is equal to

$$-4\pi\iint \{lA+mB+nC\}\,dS.$$

Therefore

$$\iint \{l(\alpha+4\pi A)+m(\beta+4\pi B)+n(\gamma+4\pi C)\}\,dS$$

$$=\iint \{la+mb+nC\}\,dS=0.$$

Since

$$\iint \{la+mb+nc\}\,dS=0$$

for all possible closed surfaces, it follows that

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0$$

at every point.

It is proved in Art. 191, above referred to, that if a small cylinder be taken whose base is dS and height very small in comparison with the linear magnitude of dS, then the average force within this cylinder normal to dS arising from the included magnetic molecules is $-4\pi\sigma$, or in the magnetic notation

$$-4\pi(lA+mB+nC).$$

Whence it follows that if a crevasse be formed by emptying this cylinder of the included molecules, the average force within the cylinder normal to dS arising from the rest of the field is

$$l(\alpha+4\pi A)+m(\beta+4\pi B)+n(\gamma+4\pi C),$$

in other words, this force is the magnetic induction as above defined.

It will be observed that the magnetic induction components satisfy the no-convergence condition

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0,$$

but that the magnetic force components α, β, γ do not generally do so; on the other hand, α, β, and γ are always derived from a potential, but a, b, and c are not so, unless the components of magnetisation A, B, and C are so derived.

The magnetic induction and magnetic force are the same in all regions devoid of magnetic matter.

311.] Since the flux of magnetic induction is zero for any closed surface whatever, this flux must be the same through all surfaces bounded by the same closed curve, and therefore must be equal to a line integral taken round the curve.

Let l, m, n be direction-cosines of the normal at any point P of a surface S bounded by the closed curve s, and let a, b, c be the components of magnetic induction at P. Then, if F, G, H be vector functions of the coordinates ξ, η, ζ of any point in S which satisfy the conditions

$$a=\frac{dH}{d\eta}-\frac{dG}{d\zeta},\quad b=\frac{dF}{d\zeta}-\frac{dH}{d\xi},\quad c=\frac{dG}{d\xi}-\frac{dF}{d\eta},$$

we know, by Art. 271, that the line integral

$$\int\left(F\frac{d\xi}{ds}+G\frac{d\eta}{ds}+H\frac{d\zeta}{ds}\right)ds$$

taken round the curve s is equal to the surface integral

$$\iint(la+mb+nc)\,dS.$$

The quantities F, G, H determined by the above written equations are called the components of the *Vector potential of magnetic induction*, and sometimes also the components of the *Magnetic momentum*.

312.] In Arts. 285, 286 we have found the expression for the ordinary or scalar potential V of any given magnetic field, and we now proceed to do the same for the components F, G, H of the vector potential. This involves the solution of the simultaneous differential equations

$$\frac{dH}{d\eta}-\frac{dG}{d\zeta}=-\frac{dV}{d\xi}+4\pi A,\quad \frac{dF}{d\zeta}-\frac{dH}{d\xi}=-\frac{dV}{d\eta}+4\pi B,$$
$$\frac{dG}{d\xi}-\frac{dF}{d\eta}=-\frac{dV}{d\zeta}+4\pi C.$$

The solution is

$$F=\iiint\left(B\frac{d}{dz}-C\frac{d}{dy}\right)\frac{1}{r}\,dxdydz,\quad G=\iiint\left(C\frac{d}{dx}-A\frac{d}{dz}\right)\frac{1}{r}\,dxdydz,$$
$$H=\iiint\left(A\frac{d}{dy}-B\frac{d}{dx}\right)\frac{1}{r}\,dxdydz,$$

where r is the distance between the point ξ, η, ζ at which F, G, H are to be found and the element $dxdydz$, and the integrals are taken over all space.

For remembering that

$$\frac{d}{d\xi}\frac{1}{r}=-\frac{d}{dx}\frac{1}{r},$$

with similar relations for η and y, ζ and z, we have with these values of F, G, and H,

$$\frac{dH}{d\eta}-\frac{dG}{d\zeta}=\frac{d}{d\eta}\iiint B\frac{d}{d\xi}\frac{1}{r}\,dxdydz-\frac{d}{d\eta}\iiint A\frac{d}{d\eta}\frac{1}{r}\,dxdydz$$
$$+\frac{d}{d\zeta}\iiint C\frac{d}{d\xi}\frac{1}{r}\,dxdydz-\frac{d}{d\zeta}\iiint A\frac{d}{d\zeta}\frac{1}{r}\,dxdydz.$$

Also, by Art. 286, if V be the scalar magnetic potential,

$$\frac{dV}{d\xi}=-\frac{d}{d\xi}\iiint\left(A\frac{d}{d\xi}+B\frac{d}{d\eta}+C\frac{d}{d\zeta}\right)\frac{1}{r}\,dxdydz.$$

Therefore

$$\frac{dH}{d\eta}-\frac{dG}{d\zeta}+\frac{dV}{d\xi}=\frac{d}{d\eta}\iiint B\frac{d}{d\xi}\frac{1}{r}\,dxdydz-\frac{d}{d\xi}\iiint B\frac{d}{d\eta}\frac{1}{r}\,dxdydz$$
$$+\frac{d}{d\zeta}\iiint C\frac{d}{d\xi}\frac{1}{r}\,dxdydz-\frac{d}{d\xi}\iiint C\frac{d}{d\zeta}\frac{1}{r}\,dxdydz$$
$$-\frac{d}{d\xi}\iiint A\frac{d}{d\xi}\frac{1}{r}\,dxdydz-\frac{d}{d\eta}\iiint A\frac{d}{d\eta}\frac{1}{r}\,dxdydz-\frac{d}{d\zeta}\iiint A\frac{d}{d\zeta}\frac{1}{r}\,dxdydz$$

Here the integrals in the right-hand member are extended throughout all space, including the point from which r is measured.

We may consider them as divided into two parts, (1) for all space outside of an infinitely small sphere described about the

point in question as centre, (2) for the space within that small sphere. Then in forming the integral (1) for the external space we may differentiate under the integral sign; and this causes the right-hand member to vanish, because in this case, the point from which r is measured not being included in the limits of integration, $\nabla^2 \frac{1}{r} = 0$ for every point. The external space therefore gives

$$\frac{dH}{d\eta} - \frac{dG}{d\zeta} + \frac{dV}{d\xi} = 0.$$

Secondly, for the space within the infinitely small sphere we may, if A, B, C be continuous functions, put them outside of the sign of integration as constants. Then the first two lines of the right-hand member of our equation vanish by symmetry: and the third line becomes $4\pi A$. Hence, the integration for the space within the small sphere gives

$$\frac{dH}{d\eta} - \frac{dG}{d\zeta} + \frac{dV}{d\xi} = 4\pi A.$$

And combining the two results we have for all space

$$\frac{dH}{d\eta} - \frac{dG}{d\zeta} + \frac{dV}{d\xi} = 4\pi A,$$

or $$\frac{dH}{d\eta} - \frac{dG}{d\zeta} = a + 4\pi A = a.$$

Similarly, $$\frac{dF}{d\zeta} - \frac{dH}{d\xi} = b,$$

$$\frac{dG}{d\xi} - \frac{dF}{d\eta} = c.$$

313.] If A, B, and C are discontinuous at the point considered, we may obtain the same result as follows

$$H = \iiint \left(B\frac{d}{d\xi} - A\frac{d}{d\eta}\right)\frac{1}{r}\, dx\, dy\, dz$$
$$= \iint \frac{lB - mA}{r}\, dS - \iiint \frac{1}{r}\left(\frac{dB}{d\xi} - \frac{dA}{d\eta}\right) dx\, dy\, dz,$$

in which the double integral is over every surface of discontinuity of A and B, and throughout the triple integral B and A are continuous. Treating G and also V in the same manner, we obtain

$$\frac{dH}{d\eta}-\frac{dG}{d\zeta}+\frac{dV}{d\xi}=\frac{d}{d\eta}\iint\frac{lB-mA}{r}\,dS-\frac{d}{d\zeta}\iint\frac{nA-lC}{r}\,dS$$
$$-\frac{d}{d\xi}\iint\frac{lA+mB+nC}{r}\,dS$$
$$-\frac{d}{d\eta}\iiint\frac{1}{r}\left(\frac{dB}{d\xi}-\frac{dA}{d\eta}\right)dx\,dy\,dz-\frac{d}{d\zeta}\iiint\frac{1}{r}\left(\frac{dC}{d\xi}-\frac{dA}{d\zeta}\right)dx\,dy\,dz$$
$$-\frac{d}{d\xi}\iiint\frac{1}{r}\left(\frac{dA}{d\xi}+\frac{dB}{d\eta}+\frac{dC}{d\zeta}\right)dx\,dy\,dz.$$

The integral taken throughout the space outside of the small sphere enclosing the point considered is zero by Art. 312, because we may perform the differentiations under the integral sign. When we integrate throughout the small sphere the triple integrals in the second member vanish, because the quantities under the integral sign are finite. Of the double integrals the first represents the force in y, due to a distribution of density $lB-mA$ over the surface of discontinuity passing through the point considered. That is, $-4\pi m(lB-mA)$. Treating the other double integrals in the same way, we find for the sum of the three

$$-4\pi m(lB-mA)+4\pi n(nA-lC)+4\pi l(lA+mB+nC)=4\pi A.$$

Therefore, as before,

$$\frac{dH}{d\eta}-\frac{dG}{d\zeta}+\frac{dV}{d\xi}=4\pi A,$$

or
$$\frac{dH}{d\eta}-\frac{dG}{d\zeta}=a+4\pi A=a,$$
$$\&c. = \&c.$$

314.] When the magnetisation is lamellar A, B, and C are derivable from a potential ϕ, and therefore in this case a, b, and c are so likewise.

Referring to Art. 304, we see that in such a mass the quantity in that Article called Ω, or

$$\iint\frac{\phi\cos\theta}{r^2}\,dS,$$

is the potential of magnetic induction, and that the components of this induction are

$$-\frac{d\Omega}{d\xi}, \quad -\frac{d\Omega}{d\eta}, \quad -\frac{d\Omega}{d\xi}.$$

In this case

$$F = \iiint \left\{ \frac{d\phi}{dy}\frac{d}{dz} - \frac{d\phi}{dz}\cdot\frac{d}{dy} \right\} \frac{dx\,dy\,dz}{r}$$

$$= \iint \phi \left(m\frac{d}{dz} - n\frac{d}{dy} \right) \frac{dS}{r}$$

or

$$= \iint \left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz} \right) \frac{dS}{r}.$$

The quantities denoted by F, G, H in Art. 301, are the components of vector potential for a uniform magnetic shell, and, as there stated, for such a shell

$$F = \phi \int \frac{1}{r}\frac{dx}{ds}\,ds, \quad G = \phi \int \frac{1}{r}\frac{dy}{ds}\,ds, \quad H = \phi \int \frac{1}{r}\frac{dz}{ds}\,ds,$$

the line integration being taken round the shells contour.

These results might have been deduced from the expressions for F, G, and H just found for any lamellar mass.

Of Induced Magnetism.

315.] Hitherto we have treated of magnets and magnetic molecules in their mechanical relations only, considering magnetisation as an invariable quantity without regard to the means by which it can be produced, altered, or destroyed. In nature no such thing as an invariably magnetised body exists. Magnetisation is always changing, and in particular the magnetisation of any substance generally changes with the state of the magnetic field in which the substance is placed. Magnetisation is said to be *induced* in it by variation of the field. Generally, a piece of iron tends to assume magnetisation if originally unmagnetised, or additional magnetisation if partially magnetised, in a direction opposite to that of the field, that is, in such a direction as to diminish the magnetic potential of the field. If the field were one of electric instead of magnetic force, and the magnetisable substance a conductor, it would become polarised in that direction, and the polarization would be pro-

portional to the inducing force. In Poisson's theory of induced magnetism this is the action which ensues in the molecules of magnetisable masses when brought into any magnetic field. The molecules become polarised to a degree proportioned to the magnetic force. Hence, it follows that the mathematical treatment of such a magnetisable medium would exactly resemble that of the dielectric medium considered above in Chapters X and XI, the magnetisable molecules taking the place of the small conductors of that chapter. So that if α, β, γ were the components of total magnetic force of the field at any point, we should have an additional magnetisation in the neighbourhood of that point arising from induction whose components were, in the notation of that chapter, $Q\alpha$, $Q\beta$, and $Q\gamma$ respectively, the symbol κ being generally used for Q in Poisson's notation [1].

The quantity $1+4\pi\kappa$ is, therefore, in all respects analogous to that represented by K or $1+4\pi Q$ in the chapters referred to, and it is in Poisson's notation generally denoted by μ. Further, A, B, and C, the components of induced magnetisation, are respectively equal to $\kappa\alpha$, $\kappa\beta$, and $\kappa\gamma$.

It follows from the results arrived at in the aforesaid chapters, that the magnetic potential at any point in a magnetisable mass, in any magnetic field, is $\frac{1}{\mu}$ of the potential at the same point in air or vacuum, and therefore that in comparing two media with different values of μ, the intensities of the fields arising from similar magnetic systems vary inversely as μ, that is, α, β, and γ, the forces derived from magnetised molecules, vary inversely as μ. On the other hand, the vector whose components are $\mu\alpha$, $\mu\beta$, $\mu\gamma$ is always independent of μ.

When the magnetisation of the mass arises entirely from induction, the last mentioned vector is the magnetic induction, and in this case the magnetic induction at any point in any medium due to any given magnetic distribution is independent of μ, and whatever be the changes of medium, the flux of

[1] It is assumed in the text that we are dealing with iron, by far the most important of magnetisable substances. In certain substances induced magnetism is of the opposite sign to that stated in the text. Such substances are called diamagnetic. Iron, and substances which behave like it are paramagnetic.

magnetic induction over every closed surface is zero, and the magnitudes F, G, H, are independent of μ.

In all cases the flux of the vector $\mu\alpha$, $\mu\beta$, $\mu\gamma$ over any closed surface is equal to the algebraic sum of the magnetism within the surface.

316.] If a homogeneous mass without magnetisation, but capable of being magnetised by induction, be placed in a magnetic field, the magnetisation which it assumes is, according to this theory, lamellar and solenoidal.

For let V be the magnetic potential, including as well that of the field as that of the induced magnetism. Then we have at every point in the mass

$$A = -\mu \frac{dV}{dx},$$
$$B = -\mu \frac{dV}{dy},$$
$$C = -\mu \frac{dV}{dz}.$$

And therefore since the mass is homogeneous and μ constant, A, B, C are derived from a potential $-\mu V$, and the magnetisation is lamellar. Again, if ρ be the density of magnetic matter within the mass, $\nabla^2 V + 4\pi\rho = 0$. Also, as shown above,

$$\rho = -\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right),$$
$$= \mu\nabla^2 V, \quad \text{or} \quad -\mu\nabla^2 V + \rho = 0;$$

whence it follows that $\rho = 0$, and

$$\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} = 0,$$

or the magnetisation is solenoidal.

A case of a lamellar mass is conceivable in which the force, due to the mutual attraction of the faces of every magnetic shell into which the substance is divided, is always equal and opposite to the separating force to which the magnetisation is due, so that $\alpha = 0$, $\beta = 0$, $\gamma = 0$ at every point. According to the experiments of Thalen (Maxwell, 430) this condition is

very nearly reached by soft iron, for which $\frac{\mu-1}{4\pi} = 32$, and therefore $\mu = 128\pi + 1$.

317.] On this theory it follows that to any problem in induced magnetism there corresponds a problem in specific inductive capacity, and any such problem may be investigated on the principles developed in Chap. XI, with the substitution of μ for the symbol K of that Chapter, where $\mu = 1 + 4\pi\kappa$.

If V be the known potential at any point of the *given* magnetism in the field, and V' that of the *induced* magnetism, and we confine our attention to the case of isotropic media, the equations for the determination of the unknown quantity V' in terms of the given quantity V are of the form

$$\frac{d}{dx}\mu\left(\frac{dV}{dx}+\frac{dV'}{dx}\right)+\frac{d}{dy}\mu\left(\frac{dV}{dy}+\frac{dV'}{dy}\right)+\frac{d}{dz}\mu\left(\frac{dV}{dz}+\frac{dV'}{dz}\right)+4\pi\rho=0,$$

throughout regions wherein μ is either invariable or continuously variable, and

$$\mu\frac{d}{d\nu}(V+V')+\mu'\frac{d}{d\nu'}(V+V')+4\pi\sigma=0,$$

over surfaces separating the media at which μ changes discontinuously from μ to μ', where ρ and σ are the volume and superficial densities of any given fixed magnetism in the neighbourhood of the point. In the case of a single magnetisable mass bounded by a given surface and placed in air or a medium for which μ is unity, and in a given external magnetic field, and if there be no fixed magnetism in the mass, the above equations are reduced to

$$\nabla^2 V' = 0,$$

and

$$\mu\frac{d}{d\nu}(V+V')+\frac{d}{d\nu'}(V+V')=0.$$

The last equation becomes, by the substitution of $1+4\pi\kappa$ for μ,

$$(1+4\pi\kappa)\frac{dV'}{d\nu}+\frac{dV'}{d\nu'}+4\pi\kappa\frac{dV}{d\nu}=0,$$

because

$$\frac{dV}{d\nu'}+\frac{dV}{d\nu}=0,$$

and it is more generally written in this form.

318.] For all cases of concentric spherical boundaries it is easy to determine V' in suitable spherical harmonic functions, when V has been so expressed, with the common spherical centre as origin. The particular case of a sphere with magnetic permeability μ, surrounded by air in a field of uniform magnetic force F, parallel to x, is specially treated in Chap. XI above mentioned.

If in the results there obtained we write μ for K, we get for the resulting magnetic potential V in the space outside the sphere

$$V = -Fx + F \cdot \frac{\mu-1}{\mu+2} \frac{a^3 x}{r^3} + C,$$

and for the potential V_1 in the sphere

$$V_1 = -\frac{3Fx}{\mu+2} + C.$$

And for the superficial magnetisation of the sphere at any point

$$\sigma = \frac{3F}{4\pi} \cdot \frac{\mu-1}{\mu+2} \cdot \frac{x}{a}.$$

The method may be extended to a mass bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

surrounded by air and in a field of constant force F.

For if ϕ denote the integral

$$\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

where λ is a function of x, y, z determined by the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we know that $x\dfrac{d\phi}{da^2}$ is the x-component at x, y, z of the attraction of a mass of density unity bounded by the ellipsoid, and therefore it satisfies the conditions of vanishing at infinity, and

$$\nabla^2\left(x\frac{d\phi}{da^2}\right) = 0$$

at all external points.

If ϕ_0 denote the value of ϕ when $\lambda = 0$, that is to say, for any point on the ellipsoid, it follows that the function

$$\frac{x\dfrac{d\phi}{da^2}}{\dfrac{d\phi_0}{da^2}}$$

satisfies the above conditions, and is equal to x at the surface of the ellipsoid. Also at the surface,

$$\frac{d}{d\nu}\left(\frac{x\dfrac{d\phi}{da^2}}{\dfrac{d\phi_0}{da^2}}\right) = \varpi\frac{x}{a^2} + \frac{x}{\dfrac{d\phi_0}{da^2}}\left\{\varpi\frac{x}{a^2}\frac{d}{dx} + \varpi\frac{y}{b^2}\frac{d}{dy} + \varpi\frac{z}{c^2}\frac{d}{dz}\right\}\frac{d\phi}{da^2}$$

$$= \frac{\varpi x}{a^2} + \frac{\varpi x}{\dfrac{d\phi_0}{da^2}}\left\{\frac{x}{a^2}\cdot\frac{d\lambda}{dx} + \frac{y}{b^2}\frac{d\lambda}{dy} + \frac{z}{c^2}\frac{d\lambda}{dz}\right\}\frac{d}{d\lambda}\cdot\frac{d\phi}{da^2},$$

where ϖ is the perpendicular from the centre on the tangent plane at x, y, z.

Also $$\frac{d}{d\lambda}\cdot\frac{d\phi}{da^2} = \frac{1}{2}\cdot\frac{1}{a^3bc} \text{ when } \lambda = 0.$$

And from (1), where $\lambda = 0$,

$$\frac{d\lambda}{dx} = \frac{2x}{a^2}\varpi^2, \quad \frac{d\lambda}{dy} = \frac{2y}{b^2}\varpi^2, \quad \frac{d\lambda}{dz} = \frac{2z}{c^2}\varpi^2.$$

Therefore $$\frac{d}{d\nu}\left(\frac{x\dfrac{d\phi}{da^2}}{\dfrac{d\phi_0}{da^2}}\right) = \frac{\varpi x}{a^2} = \frac{\varpi x}{a^2}\cdot\frac{1}{abc\dfrac{d\phi_0}{da^2}}. \quad . \quad . \quad . \quad . \qquad (2)$$

Therefore we obtain a solution by assuming for the potential V of the external space

$$V = -Fx\left\{1 - \frac{\dfrac{d\phi}{da^2}}{\dfrac{d\phi_0}{da^2}}\right\} + \frac{Ax\dfrac{d\phi}{da^2}}{\dfrac{d\phi_0}{da^2}} + C,$$

and for the potential, V_1, of the internal space

$$V_1 = Ax + C,$$

where A is a constant to be determined.

Again, on the surface

$$\frac{dV}{d\nu} - \mu\frac{dV_1}{d\nu} = 0.$$

The last equation gives us, by means of (2),

$$\mu A = \frac{F}{abc\frac{d\phi_0}{da^2}} + A + \frac{A}{abc\frac{d\phi_0}{da^2}},$$

$$\text{or} \quad \left\{(\mu-1)abc\frac{d\phi_0}{da^2} - 1\right\} A = +F,$$

which determines A.

When $\qquad a = b = c, \quad \frac{d\phi_0}{da^2} = -\frac{1}{3a^3};$

the ellipsoid then becomes a sphere, and

$$A = -\frac{3F}{\mu+2}$$

as before.

The superficial magnetisation is

$$\frac{1}{4\pi}(\mu-1)\frac{dV_1}{d\nu} = -\frac{F}{4\pi}\frac{\mu-1}{(\mu-1)abc\frac{d\phi_0}{da^2} - 1}\varpi\frac{x}{a^2}.$$

If the given field had been one of constant force whose components were F, G, H, then the internal field would also have been one of constant force whose components were

$$\frac{-F}{(\mu-1)abc\frac{d\phi_0}{da^2} - 1}, \quad \frac{-G}{(\mu-1)abc\frac{d\phi_0}{db^2} - 1}, \quad \text{and} \quad \frac{-H}{(\mu-1)abc\frac{d\phi_0}{dc^2} - 1}.$$

And the superficial magnetism at any point would be

$$-\frac{(\mu-1)\varpi}{4\pi}\left\{\frac{F}{(\mu-1)abc\frac{d\phi_0}{da^2} - 1}\cdot\frac{x}{a^2} + \frac{G}{(\mu-1)abc\frac{d\phi_0}{db^2} - 1}\cdot\frac{y}{b^2} + \frac{H}{(\mu-1)abc\frac{d\phi_0}{dc^2} - 1}\cdot\frac{z}{c^2}\right\}.$$

319.] We may apply a similar treatment to the case of a shell bounded by concentric spherical surfaces with radii a and b, and situated in air in a field of uniform force F.

Let V, V', and V'' be the potentials in the external space, in the substance of the shell, and within the hollow respectively; then it is clear that

$$V = -Fx\left(1-\frac{a^3}{r^3}\right)+\frac{Aa^3}{r^3}x+Q,$$

$$V' = Bx\left(1-\frac{a^3}{r^3}\right)+Cx\left(1-\frac{b^3}{r^3}\right)+Q,$$

$$V'' = Dx+Q,$$

where Q is a constant, will at all points satisfy the condition $\nabla^2 V = 0$.

Also V at infinity becomes $-Fx+Q$, and V' and V'' are everywhere finite.

If $A = C\left(1-\frac{b^3}{a^3}\right)$, and $D = B\left(1-\frac{a^3}{b^3}\right)$, the values of the potential will be everywhere continuous.

The surface conditions require that

$$\frac{dV}{dr_{r=a}} = \mu\frac{dV'}{dr_{r=a}}, \quad \text{and } \mu\frac{dV'}{dr_{r=b}} = \frac{dV''}{dr_{r=b}},$$

whence we get

$$3\mu B + C\left(2+\mu+2(\mu-1)\frac{b^3}{a^3}\right) = -3F,$$

$$3\mu C\frac{b^3}{a^3}+B\left(1+2\mu+(\mu-1)\frac{b^3}{a^3}\right) = 0,$$

or eliminating and reducing,

$$B\left\{9\mu+2(\mu-1)^2\left(1-\frac{b^3}{a^3}\right)\right\} = \frac{9\mu F\frac{b^3}{a^3}}{1-\frac{b^3}{a^3}}.$$

Also $$D = -B\frac{a^3}{b^3}\left(1-\frac{b^3}{a^3}\right).$$

Therefore $$D = -\frac{9\mu F}{9\mu+2(\mu-1)^2\left(1-\frac{b^3}{a^3}\right)}.$$

And the hollow is a field of uniform magnetic force whose value is

$$\frac{9\mu}{9\mu+2(\mu-1)^2\left(1-\frac{b^3}{a^3}\right)}.$$

320.] A similar method may be applied to the case of a shell

bounded by concentric and confocal ellipsoidal surfaces surrounded by air in a field of uniform force parallel to one of the axes.

Let the equations of the outer and inner ellipsoids respectively be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \text{and} \quad \frac{x^2}{a^2-\lambda_1}+\frac{y^2}{b^2-\lambda_1}+\frac{z^2}{c^2-\lambda_1}=1,$$

and let ϕ and ϕ' denote the integrals

$$\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}$$

and $$\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2-\lambda_1+\lambda)(b^2-\lambda_1+\lambda)(c^2-\lambda_1+\lambda)}},$$

and ϕ_0 and ϕ'_0 the corresponding integrals for $\lambda=0$.

The value of λ for any point x, y, z in the respective integrals being determined by the equations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \text{ and } \frac{x^2}{a^2-\lambda_1+\lambda}+\frac{y^2}{b^2-\lambda_1+\lambda}+\frac{z^2}{c^2-\lambda_1+\lambda}=1.$$

It will be observed that

$$\int_x^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} = \int_{\lambda_1+x}^\infty \frac{d\lambda}{\sqrt{(a^2-\lambda_1+\lambda)(b^2-\lambda_1+\lambda)(c^2-\lambda_1+\lambda)}},$$

and therefore that at any point in space $\phi=\phi'$, provided that in forming ϕ we measure λ from the outer, and in forming ϕ' we measure λ from the inner ellipsoid.

Then the values of the aforesaid potentials V, V', and V'' for the external space, the shells substance, and the hollow will be given by the following equations

$$V=-Fx\left\{1-\frac{\frac{d\phi}{da^2}}{\frac{d\phi_0}{da^2}}\right\}+Ax\frac{\frac{d\phi}{da^2}}{\frac{d\phi_0}{da^2}},$$

$$V'=Bx\left\{1-\frac{\frac{d\phi}{da^2}}{\frac{d\phi_0}{da^2}}\right\}+Cx\left\{1-\frac{\frac{d\phi'}{d(a^2-\lambda_1)}}{\frac{d\phi'_0}{d(a^2-\lambda_1)}}\right\},$$

$$V''=Dx,$$

where
$$A = C\left\{1 - \frac{\dfrac{d\phi'}{d(a^2-\lambda_1)}}{\dfrac{d\phi'_0}{d(a^2-\lambda_1)}}\right\},$$
$$D = B\left\{1 - \frac{\dfrac{d\phi'}{d(a^2-\lambda_1)}}{\dfrac{d\phi_0}{da^2}}\right\},$$
with the superficial conditions at the outer and inner surfaces
$$\frac{dV}{d\nu} = \mu\frac{dV'}{d\nu} \quad \text{and} \quad \mu\frac{dV'}{d\nu} = \frac{dV''}{d\nu}.$$

At the outer surface
$$\frac{dV}{d\nu} = \frac{Fx\varpi}{a^2}\cdot\frac{1}{abc\dfrac{d\phi_0}{da^2}} + A\frac{\varpi x}{a^2} + A\frac{\varpi x}{a^2}\cdot\frac{1}{abc\dfrac{d\phi_0}{da^2}},$$
$$\frac{dV'}{d\nu} = B\frac{x\varpi}{a^2}\cdot\frac{1}{abc\dfrac{d\phi_0}{da^2}} + C\frac{\varpi x}{a^2}\left(1 + \frac{\dfrac{d\phi_0}{da^2}}{\dfrac{d\phi'_0}{d(a^2-\lambda_1)}}\right),$$
where ϖ is the perpendicular from the centre on the tangent plane at x, y, z of the outer surface. Substituting in the first of the superficial equations and dividing by $\dfrac{\varpi x}{a^2}$ we obtain a linear equation in the constants.

Similarly, we should find that at the inner surface $\dfrac{dV'}{d\nu}$ and $\dfrac{dV''}{d\nu}$ are each divisible by $\dfrac{\varpi' x}{a^2-\lambda_1}$, where ϖ' is the perpendicular from the centre on the tangent plane at the point x, y, z of that surface, whence we should get a second linear equation—these two equations in A, B, C, D combined with the two given above give a complete solution of the problem.

321.] The theory of induced magnetism given above does not adequately explain the phenomena presented by soft iron or other magnetisable substances when placed in a magnetic field. According to the theory, the intensity of magnetisation induced should be proportional to the force, and so capable of increase without limit. And the magnetisation should immediately dis-

appear on removal of the force. Neither of these conditions is fulfilled in practice. It is found that the magnetisation actually assumed by soft iron tends, as the inducing force is increased, to a definite limit, and that it does not immediately or entirely disappear on removal of the force. It is found also that if the force pass through a complete cycle the magnetisation is always retarded in phase, as a consequence of which work is done in the cycle. Other theories have been invented to explain the actual phenomena, of which the best known is that of Weber, discussed by Maxwell, Chap. VI, Vol. II. The reader may also consult the works in footnote below*. It is not our purpose to dwell on this branch of the subject, which belongs rather to treatises on the physical properties of iron.

* Warburg, Wiedemanns Annalen XIII, p. 141; Dr. Hopkinson, Phil. Trans., Vol. CLXXVI, part II, p. 455; Professor Ewing, ibid., p. 523; Lord Rayleigh, Phil. Mag., Vol. XXII, p. 175; Mr. Bosanquet, ibid., Vol. XIX, pp. 57, 73, 333; Vol. XXII, p. 500.

CHAPTER XVIII.

MUTUAL RELATIONS OF MAGNETS AND ELECTRIC CURRENTS.

ARTICLE 322.] WE now return to the consideration of the system of two uniform magnetic shells of strengths ϕ, ϕ' respectively; and, until otherwise stated, it will be understood that we are dealing with a medium in which the magnetic permeability is unity. As above shown, the potential energy of mutual action of the two shells, that is the work which would be done in constructing the shell ϕ against the forces exerted by the shell ϕ' is

$$-\phi \iint (la' + mb' + nc')\, dS,$$

where a', b', c' are the components of magnetic induction, or, which is here the same thing, magnetic force, due to the shell ϕ', and the integration is over the shell ϕ. The surface integral represents the flux of magnetic induction of the shell ϕ' through the shell ϕ, or, as we may otherwise express it, the number of lines of magnetic induction of the shell ϕ' which pass through the shell ϕ.

The expression

$$\phi \iint (la' + mb' + nc')\, dS$$

admits of being put in several other forms of which we shall have occasion to make use; viz.

$$\phi \iint (la' + mb' + nc')\, dS = \phi\phi' \iint \frac{\cos\epsilon}{r}\, ds\, ds'$$

where ϵ is the angle between ds and ds', taken round the boundary of both shells. We shall denote the integral

$$\iint \frac{\cos\epsilon}{r}\, ds\, ds' \text{ by } M.$$

Also

$$\phi\iint(la'+mb'+nc')dS = \phi\int\left(F'\frac{dx}{ds}+G'\frac{dy}{ds}+H'\frac{dz}{ds}\right)ds$$

round the boundary of the shell ϕ, where F', G', H' are the components of vector potential of magnetic induction due to the shell ϕ'. It appears from the last expressions that the quantity of work in question is independent of the form of the surface of either shell if the bounding curve be given.

If the shells be rigidly magnetised, and if they be capable of relative motion without change of shape, they will so move as to diminish the quantity $-\phi\phi' M$, that is to increase or diminish M, according as ϕ and ϕ' have the same or opposite signs.

Exactly in the same way if there be many magnetic shells, or magnetised bodies, in the field, a rigidly magnetised shell of invariable shape will, if free to move without change of shape, so move as to increase the flux of magnetic induction due to the field through its contour.

Evidently any such diminution of the potential energy has its equivalent in kinetic energy of visible motion of the shells, or in external work done.

323.] It was discovered by Oersted that the field in the neighbourhood of a closed electric current is a magnetic field. The definition of this field, usually accepted as the result of experiments, is that the magnetic field due to a uniform magnetic shell at any point not within its substance is the same as that due to a certain closed electric current round the bounding curve of the shell. The direction of the current is the positive direction as defined in Art. 269, taking for the positive normal to the shells surface a line drawn from the negative towards the positive face of the shell. The strength of the current is proportional to that of the shell. When we come to treat of the units of measurement, we shall see that in a certain system, called the electromagnetic system, the strength of the shell is numerically equal to that of the current.

324.] An infinite straight line may be regarded as the edge of a plane magnetic shell, every other part of the boundary of which is infinitely distant. According to the law of equivalence above

stated, such a shell produces the same magnetic field as an electric current in the infinite straight line. Let a small bar magnet be brought into the field of the infinite shell. Such a magnet may be regarded as a magnetic shell, or aggregate of parallel magnetic shells. We might then form the integral

$$\iint \frac{\cos \epsilon}{r}\, ds\, ds',$$

for the infinite straight line with the boundary of the shell or shells composing the magnet in any given position. The bar magnet, if free to move, would tend so to place itself as to make this integral a maximum, and, according to the law of equivalence above stated, it will behave in the same way when for the infinite shell we substitute an electric current in the infinite line. Let us take the direction of the current for axis of z, and a plane through C, the centre of the magnet, for that of xy, the origin O being at the intersection of this plane with the infinite straight line. Then, first, let the axis of the magnet be constrained to lie in a radius drawn from O in the plane of xy, but be free to rotate about an axis coinciding with that of z. In this case the integral

$$\iint \frac{\cos \epsilon}{r}\, ds\, ds' = 0,$$

whatever be the length CO, and whatever angle it makes with a fixed plane through the axis of z. The magnet therefore will be acted upon by no couple tending to turn it round the axis of z. It is found that this is in fact the case.

Secondly, let the centre of the magnet be fixed, and let it be free to turn in a plane perpendicular to the current. In this case the integral is a maximum when the positive pole points towards the right of a man so standing that the current flows from his head to his feet and facing the magnet. It is found that the magnet does tend so to place itself. The experiment might therefore be regarded as confirming the law of equivalence above stated.

The experiment can also be interpreted in a somewhat different way. The magnet may be regarded as consisting of a

positive and a negative pole. And the current as exerting a force on the positive pole in the tangent to a circle drawn through the pole round the origin as centre in a plane perpendicular to the current, in the direction above indicated, and, *cæteris paribus*, an equal and opposite force on the negative pole.

Since in the first case there is no resultant couple tending to move the magnet as a whole round the origin, it follows that the moment of the force acting on the positive pole round the axis is equal and opposite to that acting on the negative pole for every position of the magnet. Whence it is inferred that the force on a pole due to the current in the direction of the tangent varies inversely as the distance of the pole from the current. For a magnet of invariable shape the experiment admits indifferently of either interpretation.

325.] As the magnetic field due to a closed electric current is the same as that due to the equivalent magnetic shell, it follows that the mechanical effect of the field on the conductors carrying the current is the same as its mechanical effect on the shell. That is, if the electric current i be maintained constant, the circuit, if rigid, tends to move so as to increase or diminish the coefficient M, that is the flux of magnetic induction through it, exactly as the equivalent shell would do if rigidly magnetised.

If q be any generalised coordinate on which the value of M, or

$$\iint \frac{\cos\epsilon}{r}\, ds\, ds',$$

depends, the mechanical force tending to increase q in a system of magnetic shell and electric current (the magnetisation of the shell and the electric current being both constant, and the shape of shell and circuit invariable) is $i\phi \frac{dM}{dq}$. And therefore if the system move with i and ϕ constant under its own mutual forces so as to make q become $q+\delta q$, it acquires kinetic energy of visible motion of the shell or conductors or both, or does work, equal to $i\phi \frac{dM}{dq}\delta q$.

In the corresponding case of two shells we said that this kinetic

energy, or work done, was equivalent to the diminution of the potential energy of position of the two shells caused by the motion. In the case of circuit and shell now under consideration, it is true that the forces are derived from $i\phi M$ as from a potential. Nevertheless, to ascribe to the system potential energy of position would not be a complete account of the phenomena. Because, as we shall see later, the motion involves an increased expenditure of chemical energy in the battery to maintain the current constant over and above what would have been necessary for this purpose had the system remained at rest. And the external work done by the system has its exact equivalent in the additional chemical energy spent in the battery.

326.] If for the two shells we substitute the two equivalent electric circuits with currents i and i', their mutual mechanical action, assuming the currents to be maintained constant, is the same as that of the shells. They tend to move so as to increase the quantity

$$i'\iint\frac{\cos\epsilon}{r}\,ds\,ds', \text{ or } ii'M.$$

Any variation of $ii'M$ has its equivalent in external work done or kinetic energy of visible motion acquired by the conductors. But, as we shall see later, the motion of the conductors with constant currents involves in this case an increased expenditure of chemical energy *in each of the two circuits* equal to the external work done. So that in the whole chemical energy is drawn upon to twice the amount of external work done in addition to the heat generated by resistance in the circuits.

327.] The equivalence of electric currents and the corresponding magnetic shells affords a measure of electric quantity differing from that employed in Part I, Chap. IV. For instance, two infinite parallel magnetic shells bounded by two parallel straight lines attract each other if magnetised in the same direction. And therefore two infinite parallel straight currents, if in the same direction, attract each other with a force proportional to the product of their intensities. Hence we might define the unit of electricity theoretically as the quantity which must pass through a section of either current in unit time, in order that the currents

being at unit distance apart the force on unit length of either may be unit force. See Chap. XX.

328.] If any closed curve S be drawn in the field of a magnetic shell, the line integral of magnetic force round S must be zero, whether S cut the shell or not, because the force is derived from a single valued potential. If, however, the curve cut the shell once, and so embrace the bounding curve of the shell, we may take two points P and P', in the curve infinitely near one another but on opposite sides of the shell, and the potential at P will be $2\pi\phi$, and $-2\pi\phi$ at P', where ϕ is the strength of the shell. Hence the work done by the magnetic force on a unit magnetic pole in passing from P to P' always outside of the shell is $4\pi\phi$, and in passing from P' to P through the shell $-4\pi\phi$. If now for the magnetic shell we substitute the equivalent closed electric current i, we see that the line integral of magnetic force on a unit pole round a closed curve S, not embracing the current, is the same as in the former case, and therefore zero. But if the closed curve S embrace the current, inasmuch as no part of S now corresponds to the space between the faces of the shell, the magnetic force is at all points of S in the same direction round S, and its line integral on a unit pole round S must be $+4\pi i$, or $-4\pi i$, according to the direction taken.

If the electric current i were an invariable property of the circuit, this result would be contrary to the conservation of energy. But in fact the electric current can only be maintained by a continuous expenditure of energy in a battery or otherwise, the amount of which per unit of time is altered during any time variation of the magnetic field in which the circuit finds itself; and we shall see later that the passage of a magnetic pole round the closed curve S embracing the current must, if the current be maintained constant, involve the expenditure of an amount of chemical energy in the battery equal to $-4\pi i$ or $+4\pi i$ as the case may be, over and above what would otherwise have been expended in maintaining the constant current i against the resistance of the circuit.

329.] It thus appears that the magnetic potential due to a closed constant current i, if defined with reference to mechanical

forces only, may have any one of an infinite number of values differing from one another by $4\pi i$. It is the work done in bringing a unit pole from an infinite distance to the point considered by any path arbitrarily chosen, and differs by $4\pi i$ for every time that this path embraces the current.

In the case of the infinite straight current already treated, the potential is $4\pi i \tan^{-1}\frac{y}{x}$, where the plane of x, y is perpendicular to the current and the origin in the current, and we pass from the axis of x to that of y by turning in the positive direction, the current being in the direction of negative z.

330.] The effect of variation of the magnetic permeability μ between one uniform medium and another will be considered in Chapter XIX. It is sufficient here to point out that the line integral of magnetic force taken round a closed current i in the positive direction is $4\pi i$, whatever be the nature of the medium. Whence it follows from the relation between magnetic force and magnetic induction in a field of magnetisable matter (Art. 315), that the line integral of magnetic induction round the same closed current is $4\pi\mu i$.

331.] Now let I denote the current i referred to unit of area, so that if a be the transverse section of the tube through which the current i flows, $Ia = i$.

Let u, v, w be the components of I. Then we have

$$I\frac{dx}{ds} = u, \text{ \&c.}$$

Let I', u', v', w' have similar meanings for the current i'. Then

$$\begin{aligned} ii'\frac{\cos\epsilon}{r}\,ds\,ds' &= IaI'a'\frac{\cos\epsilon}{r}\,ds\,ds' = IaI'a'\frac{\frac{dx}{ds}\frac{dx'}{ds'}+\frac{dy}{ds}\frac{dy'}{ds'}+\frac{dz}{ds}\frac{dz'}{ds'}}{r}\,ds\,ds' \\ &= \frac{uu'+vv'+ww'}{r}\,a\,ds\,a'ds'. \end{aligned}$$

And therefore

$$ii'\iint\frac{\cos\epsilon}{r}\,ds\,ds' = \iiiiii\frac{uu'+vv'+ww'}{r}\,dx\,dy\,dz\,dx'\,dy'\,dz'$$

taken over both currents.

Now

$$-ii'\iint\frac{\cos\epsilon}{r}\,ds\,ds'$$

is the mechanical work which would be done in bringing the two circuits with constant currents i, i' from an infinite distance to their actual position. Therefore, also,

$$-\iiiint\!\!\iint\frac{uu'+vv'+ww'}{r}\,dx\,dy\,dz\,dx'\,dy'\,dz'$$

represents this same amount of mechanical work.

332.] We should obtain consistent results, so far as closed circuits with constant currents are concerned, if we assumed that the mechanical work done in bringing the two *elementary currents*

$$u\,dx\,dy\,dz,\quad u'\,dx'\,dy'\,dz'$$

from an infinite distance to their actual position is

$$-\frac{uu'}{r}\,dx\,dy\,dz\,dx'\,dy'\,dz',$$

and so on for every pair of parallel elementary currents, but that between two mutually perpendicular elements no work is done. Or, which is the same thing, we should obtain consistent results so far as closed circuits are concerned, if we assumed the following law of force between elementary currents, viz. that two parallel elementary currents if in the same direction attract, and if in opposite directions repel, each other with a force varying directly as the product of their intensities, and inversely as the square of the distance between them, but that mutually perpendicular elementary currents have no mutual action.

333.] We have found

$$-\iiiint\!\!\iint\frac{uu'+vv+ww'}{r}\,dx\,dy\,dz\,dx'\,dy'\,dz'$$

to be the amount of mechanical work required to bring two closed circuits with constant currents i, i' from an infinite distance to their actual position. Evidently, the work done in the case of three or more closed circuits will be the sum of a number of expressions of this form for each pair of circuits. Now, any single closed current may be regarded as the limit

of a number of similar and parallel closed currents made to coincide with each other, and on that principle we might calculate the mechanical work required to construct it. For

$$-\frac{1}{2}\iiiint\!\!\iint \frac{uu'}{r}\,dxdydz\,dx'dy'dz'$$

is the mutual potential energy of two masses of volume density u and u' respectively, and is finite if u and u' be finite, even if the two masses occupy the same space. If, therefore, I, u, v, w, the currents referred to unit area, be finite, the mechanical work done in bringing n closed circuits, each with current $\frac{i}{n}$, from an infinite distance to coincide with one another is finite.

Further, it consists of $\frac{n\,.\,\overline{n-1}}{2}$ terms, each of the form

$$-\frac{i^2}{n^2}\iint\frac{\cos\epsilon}{r}\,dsds',$$

and therefore, when n is infinite, it is independent of n. We will denote the limiting value of this expression for a given circuit or closed curve when n is infinite by $-\frac{1}{2}Li^2$.

The whole work done in constructing the closed current i will include, in addition to the above $\frac{n\,.\,\overline{n-1}}{2}$ terms, n other terms, each representing the work done in creating the closed current $\frac{i}{n}$ in its own field. We cannot assert that the expression $-\frac{1}{2}Li^2$ represents the whole work done in creating the current, without asserting that the sum of these n terms vanishes when n is infinite, compared with that of the other $\frac{n\,.\,\overline{n-1}}{2}$ terms, an assertion which may be precarious.

If this assumption can be made, the whole mechanical work done in constructing a system of two closed circuits, is

$$-\tfrac{1}{2}(L_1i_1^2+2Mi_1i_2+\tfrac{1}{2}L_2i_2^2).$$

Now, L_1 and L_2 are essentially positive, and it is not difficult to shew that $L_1L_2>M^2$.

It follows, therefore, that the mechanical work done in con-

structing the system of two closed circuits with constant currents is essentially negative.

We may call the above expression the potential energy of the two circuits, provided it be always understood that it relates to mechanical work only.

334.] In the case of a uniform magnetic shell of strength ϕ, we saw (Art. 307) that the potential self-energy of the shell is not

$$-\frac{\phi^2}{2}\iint\frac{\cos\epsilon}{r}\,ds\,ds',$$

but is that quantity increased by

$$2\pi\phi\iint I\,dS,$$

and we know that the potential energy of any magnetic system is essentially positive. In fact, since $\phi = I\,d\nu$, where $d\nu$ is the thickness of the shell, the latter term is preponderant, and the potential energy of two such shells may be written

$$\tfrac{1}{2}(\lambda_1\phi_1^2 - 2M\phi_1\phi_2 + \lambda_2\phi_2^2),$$

where λ_1 and λ_2 are no longer the limiting values of M for two shells whose contours coincide. The quantity within the bracket is in this case essentially positive.

Of the Vector Potential of Closed Electric Currents.

335.] The quantity
$$\phi\int\frac{1}{r}\frac{dx}{ds}\,ds$$
round the boundary of a magnetic shell ϕ is, as we know, the x-component of the vector potential of magnetic induction of the shell.

Similarly,
$$i\int\frac{1}{r}\cdot\frac{dx}{ds}\,ds$$
is called the x-component of vector potential of the current i, or more generally

$$\iiint\frac{u}{r}\,dx\,dy\,dz,\quad \iiint\frac{v}{r}\,dx\,dy\,dz,\quad \iiint\frac{w}{r}\,dx\,dy\,dz$$

are called the x, y, and z components of the vector potential of i, the integrals being taken over the whole space occupied by i, and if there were any number of closed currents in the field, the same integrals taken over the whole space occupied by those currents, are called the x, y, and z components of the vector potential of the whole system, and are denoted as before by F, G, and H respectively. This is on the assumption that $\mu = 1$. For it will be shown later, that if $\mu \neq 1$, we must take

$$F = \mu \iiint \frac{u}{r}\, dx\, dy\, dz, \text{ \&c.}$$

336.] Recurring to the two-current field, and supposing that $\mu = 1$, the quantity L_1, or

$$\iint \frac{\cos\epsilon}{r}\, ds\, ds',$$

taken for every pair of elements in the circuit of i_1, is the flux of magnetic induction of unit current in that circuit across any surface bounded by that circuit, or, as for brevity we shall say, across that circuit, and $L_1 i_1$ is the similar flux for the current i_1.

Therefore
$$L_1 i_1 = \int \left(F_1 \frac{dx}{ds} + G_1 \frac{dy}{ds} + H_1 \frac{dz}{ds} \right) ds$$

round the circuit of i_1, where F_1, G_1, H_1 are the components of vector potential at any point of that circuit arising from the current in it.

L_2 and $L_2 i_2$ have similar meanings with reference to the current i_2.

The quantity M, or

$$\iint \frac{\cos\epsilon}{r}\, ds\, ds',$$

is the flux of magnetic induction across i_1 of unit current in i_2, or the flux across i_2 of unit current in i_1, and Mi_2 is similarly the flux across i_1 of i_2 in the second circuit, and Mi_1 is the flux across i_2 of i_1 in the first circuit.

Therefore
$$Mi_2 = \int \left(F_1' \frac{dx}{ds} + G_1' \frac{dy}{ds} + H_1' \frac{dz}{ds} \right) ds$$

round i_1, where F_1', G_1', H_1' are components of vector poten-

tial at any point in i_1, arising from i_2 in the second circuit, with a similar meaning *mutatis mutandis* for Mi_1.

Now the quantity

$$\tfrac{1}{2}(L_1 i_1^2 + 2Mi_1 i_2 + L_2 i_2^2)$$

is equal to

$$\tfrac{1}{2}\{i_1(L_1 i_1 + Mi_2) + i_2(Mi_1 + L_2 i_2)\}.$$

It is therefore equal to

$$\frac{1}{2}\left\{i_1\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds + i_2\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds\right\},$$

where the first integral is taken round i_1, and F, G, and H in it are the components of vector potential of the whole field at every point in i_1, with similar meanings for the second integral.

The result may be written as

$$\frac{1}{2}\int i\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds,$$

taken over both the circuits.

Replacing i by $I\alpha$, &c., the expression may be written

$$\frac{1}{2}\iiint (Fu + Gv + Hw)\,dx\,dy\,dz,$$

and since

$$F = \iiint\frac{u'\,dx'\,dy'\,dz'}{r}, \qquad G = \iiint\frac{v'\,dx'\,dy'\,dz'}{r},$$
$$H = \iiint\frac{w'\,dx'\,dy'\,dz'}{r},$$

the expression may also be written as

$$\frac{1}{2}\iiiiiint\frac{uu' + vv' + ww'}{r}\,dx\,dy\,dz\,dx'\,dy'\,dz',$$

where the sextuple integral extends over the whole field. As above stated, if $\mu \neq 1$, F, G, H and the final expression must each be multiplied by μ.

337.] The quantities F, G, H of the last Article satisfy the condition

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0$$

at every point in a field of closed currents such as we are now considering.

For in such a field, the condition of continuity necessitates the equation

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$$

at all points where u, v, w vary continuously, and the corresponding equation

$$l(u_2-u_1)+m(v_2-v_1)+n(w_2-w_1)$$

over all surfaces of discontinuous variation of u, v, w.

Now

$$F=\mu\iiint\frac{u'}{r}dx'dy'dz', \quad \text{where } r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2};$$

whence

$$\frac{dF}{dx}=-\mu\iiint u'\frac{d}{dx'}\left(\frac{1}{r}\right)dx'dy'dz'$$

$$=\Sigma\mu\iint l(u_1'-u_2')\frac{1}{r}\,dS+\mu\iiint\frac{1}{r}\frac{du'}{dx'}dx'dy'dz'.$$

The Σ indicating the summation of the corresponding surface integrals over all the surfaces of discontinuous variation of u, and the triple integral being taken over all space of continuous variation of the same quantity.

Therefore

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=\Sigma\mu\iint\left\{l(u_1'-u_2')+m(v_1'-v_2')+n(w_1'-w_2')\right\}\frac{1}{r}dS$$

$$+\mu\iiint\frac{1}{r}\left(\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}\right)dx'dy'dz'=0.$$

by the aforesaid equations of continuity.

338.] If a, b, c are the components of magnetic induction at every point in the field, we know that

$$a=\frac{dH}{dy}-\frac{dG}{dz}, \quad b=\frac{dF}{dz}-\frac{dH}{dx}, \quad c=\frac{dG}{dx}-\frac{dF}{dy};$$

whence we get

$$\frac{dc}{dy}-\frac{db}{dz}=-\nabla^2F+\frac{d}{dx}\cdot\left(\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right).$$

But

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0,$$

$$\therefore\ \frac{dc}{dy}-\frac{db}{dz}=-\nabla^2F=4\pi\mu u.$$

Similarly,
$$\frac{da}{dz} - \frac{dc}{dx} = 4\pi\mu v,$$
$$\frac{db}{dx} - \frac{da}{dy} = 4\pi\mu w.$$

If S be a closed surface bounded by the curve s, it follows from these equations that

$$\int\left(a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds}\right)ds = 4\pi\mu\iint(lu+mv+nw)\,dS,$$

or the integral of the magnetic induction round any closed curve s is equal to the flux of current over a surface bounded by S multiplied by $4\pi\mu$.

In other words, it is the expression of the fact mentioned above, that the line integral of magnetic induction round a closed current i in any field is equal to $4\pi\mu i$.

It is in this respect that the expressions for the potential energy of a field of two shells

$$\tfrac{1}{2}(\lambda_1\phi_1^2 - 2M\phi_1\phi_1 + \lambda_2\phi_2^2)$$

differs from that for the two equivalent currents, or

$$-\tfrac{1}{2}(L_1 i_1^2 + M i_1 i_2 + L_2 i_2^2).$$

The former gives a potential at every point in space, but the latter only at such points as are free from currents.

The potential energy for a field of any number of closed currents is, by an obvious extension of the above reasoning, equal to

$$-\frac{1}{2}\iiint(Fu+Gv+Hw)\,dx\,dy\,dz,$$

where F, G, H are the components of vector potential at any point arising from the whole field, or as before to

$$-\frac{1}{2}\iiint\frac{uu'+vv'+ww'}{r}\,dx\,dy\,dz\,dx'\,dy'\,dz',$$

over the whole field.

339.] Since, as we have seen,

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0$$

at all points, it follows that the magnetic induction forms closed

tubes throughout all space, and that for any such tube the flux of induction through an orthogonal section is constant. Such an induction tube may be called *a magnetic circuit.*

At every point we have the equations $a = \mu a$, &c., so that the magnetic induction is connected with the magnetic force by the same formal relation as the electric current with the electromotive force.

If i be the aggregate of all closed or infinite electric currents embracing an induction tube or magnetic circuit, $4\pi i$ is the magnetic force in the circuit.

If A denote the magnetic induction through a section of the tube, $\frac{4\pi i}{A}$ is, by analogy to Ohm's law, called the *magnetic resistance* of the circuit.

CHAPTER XIX.

INDUCTIVE ACTION OF CURRENTS AND MAGNETS.

ARTICLE 340.] IT has been established by Oersted's experiments that the magnetic field due to any uniform magnetic shell is the same at any point not within the substance of the shell as the magnetic field due to a certain closed electric current coinciding with the boundary of the shell. The strength of this current in electro-magnetic units is, if the magnetic permeability be unity, equal to the strength of the shell, and the direction of the current is the positive direction, determined by taking for the positive normal to the shell's surface a normal drawn from the negative to the positive face of the shell (Art. 267). We shall speak of the current and shell as mutually equivalent.

It was observed by Faraday that if a closed circuit, with or without electromotive force of its own, be moved in the field of a magnet, a current is *induced* in it; or the current already existing in it is increased or diminished during the motion, notwithstanding that both the strength of the battery and the resistance of the circuit be unaltered. This induced current is reversed in direction if the motion be reversed, and increases with the velocity of the motion. It disappears rapidly by the resistance if the motion cease. There is then an electromotive force in the circuit due to the motion, which we may call *the electromotive force of induction*.

The same effect is of course produced by moving the magnet through the field of the circuit, and therefore also by variation of the strength of the magnet, because any such variation may be brought about by bringing a new magnet into the field to coincide with the existing magnet. It is produced by variation of the magnetic field in which the circuit is placed. On the other hand, the behaviour of a closed current is not affected by

the nature of the magnetic field in which it is placed, if there be no time variation of that field. From these facts, combined with those of Oersted, Helmholz and Thomson deduced the laws of induction between magnets and closed electric currents by a method founded on the conservation of energy.

341.] If a closed circuit with current I be disconnected from the battery, and the current allowed to decay in its own field, that is, not influenced by external induction, a certain quantity of heat is generated in the circuit during the decay. The circuit in virtue of the current in it has a certain intrinsic energy, which can at any time be got in the form of heat by disconnecting the wires from the battery. Let H denote this quantity of energy for the current i. Let i be the current at any instant after disconnecting. Then the heat developed in time dt at that instant is $Ri^2\,dt$, R denoting the resistance of the circuit, and since this can only be obtained at the expense of the intrinsic energy of the circuit, we have

$$\frac{dH}{dt} = -Ri^2, \quad H = \int_0^\infty Ri^2 dt.$$

For a given circuit the intrinsic energy is a function of i. It is the same in whichever direction through the circuit the current passes, and therefore contains only even powers of i. We shall therefore assume $H = \frac{1}{2}Li^2$, where L is a coefficient depending only on the form of the circuit, which we assume for the present to be invariable.

This gives $$Li\frac{di}{dt} = -Ri^2,$$

or $$\frac{di}{dt} = -\frac{R}{L}i,]$$

and if I be the initial current

$$i = Ie^{-\frac{R}{L}t},$$

and $$\int Ri^2 dt = \tfrac{1}{2}LI^2.$$

This intrinsic energy is independent of the nature of the magnetic field in which the decay takes place, so long as that field remains invariable with the time; but any time variation of the

eld would give rise to induced currents in the circuit according to Faraday's law, and would therefore alter the rate of decay of the current. If after the current has ceased the circuit be connected again with the battery, and the original current I re-established, the intrinsic energy is restored to the circuit; and this takes place at the expense of the battery. It follows that the chemical energy spent in the battery during the establishment of the current I in a constant magnetic field exceeds the heat generated in the circuit during the same process by the intrinsic energy of the circuit with the current I; and therefore including both processes, namely the decay of the current from I to zero, and its re-establishment, the field being in each case invariable during the variation of the current, the whole chemical energy spent is equivalent to the heat generated in the circuit.

342.] Now let there be any magnetic field, and as we are considering a theoretical case only, let it be due to a uniform magnetic shell of strength ϕ, which we can maintain constant or vary at pleasure. In this field let there be a circuit connected with a battery; and we shall suppose that either by varying the electromotive force of the battery, or by suitably adjusting the resistance, we can maintain the current constant, or make it vary in any way, notwithstanding the effect of induction in any motion of the circuit.

Let now ϕ, the strength of the shell, be constant, and let the circuit move with constant current I in obedience to the mutual attractive or repulsive forces between the shell and circuit from an initial position A to another position B. A certain amount of work, W, is done during this motion *by* the mutual forces. The circuit having arrived at B, let the wires be disconnected from the battery, and the current allowed to decay by resistance. Then let the wires, still disconnected, be moved back from B to A without current. This last-named motion may be effected without doing any work[1]. Then let the wires be reconnected,

[1] For although a current will be established inductively in the wires moving in the magnetic field, yet by diminishing without limit the velocity of the motion we can, owing to resistance, diminish without limit the current at every instant during the motion, and therefore the work done against the electromagnetic forces.

external work which we do against the forces, or $-\int I\phi \, dM$, is converted into heat, and the chemical energy of the battery is saved to the like extent. If E' be greater in absolute magnitude than E, the current I cannot be maintained constant unless the battery be reversed, in which case the chemical processes may be reversed, as in the case of an accumulator. In such a case the external work done is equivalent to the heat generated plus the chemical energy gained by the reversal of the processes. In all these cases if H be heat generated, C chemical energy spent, and W mechanical work done *by* the forces of the system, $C = H + W$, where W, and in case of an accumulator C, may be negative.

346.] It appears from the investigation of Chap. XVII that, when the circuit and shell move under the influence of their own mutual attractive or repulsive forces, $I\phi \dfrac{dM}{dt}$ is positive. And therefore the electromotive force due to the motion, or $-\phi \dfrac{dM}{dt}$, would, if it existed alone, produce a current in the opposite direction to I, that is a current tending to *resist* the motion. This law is called Lenz's law. It appears here as a result of the Conservation of Energy. This also appears at once from Faraday's experiments. For suppose a closed circuit without battery to be moved in any direction through the field of a constant magnet. An electric current is induced in it, which on cessation of the motion decays, and heat is generated. This heat can only be accounted for as the equivalent of mechanical work done during the motion. That is, the induced current must be such as to resist the motion by which it was induced.

347.] Secondly, let ϕ be again constant, and let the circuit move as before under the influence of the mutual attraction or repulsion of the circuit and shell; but instead of maintaining the current constant against the electromotive force due to the motion by increasing the strength of the battery, as in the former case, let the current be allowed to diminish. And let us so adjust the battery as that, i being the current at any instant, E shall be equal to Ri, or $Ei = Ri^2$.

In this case the chemical energy spent is all consumed in heating the circuit, and the mechanical work done *by* the forces cannot be done at the expense of chemical energy. It is done at the expense of the intrinsic energy of the system.

For the intrinsic energy at A is $\frac{1}{2}LI^2$. At any other point B in the supposed course the current i is less than I, and the intrinsic energy is $\frac{1}{2}Li^2$. The difference between these quantities of energy, or $\frac{1}{2}L\{I^2-i^2\}$, is the equivalent of the mechanical work gained, that is, $\frac{1}{2}L\{I^2-i^2\} = \int i\phi\, dM$.

Making $I-i = di$, we find for the relation between i and M during this process

$$Li\,di+i\phi\, dM = 0,$$

$$\text{or} \qquad L\,di+ \phi\, dM = 0.$$

We might call this process an *adiabatic* process by analogy to Thermodynamics. As in the first case we may make ϕ vary instead of M, and if M and ϕ both vary $L\,di + d(M\phi) = 0$.

348.] Thirdly, we may maintain E, the original electromotive force of the battery constant, that is $E = RI$, where I is the current with the circuit at rest. Let i be the current at any instant, then the chemical energy spent per unit of time is Ei, and the heat generated is Ri^2. We have in this case,

$$\frac{dW}{dt} = -\frac{d}{dt}(\tfrac{1}{2}Li^2)+Ei-Ri^2,$$

$$\text{that is,} \qquad i\frac{d}{dt}(M\phi) = -Li\frac{di}{dt}+Ei-Ri^2,$$

$$\text{or} \qquad \frac{d}{dt}(Li+M\phi) = E-Ri,$$

$$\text{or} \qquad E-\frac{d}{dt}\{Li+M\phi\} = Ri.$$

We may combine our results into one formula as follows. The energy drawn from the battery per unit of time over and above the equivalent of the heat generated is

$$i\frac{d}{dt}\{Li+M\phi\}.$$

If E be the electromotive force of the battery

$$E-\frac{d}{dt}\{Li+M\phi\} = Ri.$$

The only part of the kinetic energy T is electrokinetic or T_e. Therefore the Lagrangean equation

$$\frac{d}{dt}\left(\frac{dT}{d\dot{q}}\right)-\frac{dT}{dq}=Q$$

becomes in this case

$$-\frac{dT_e}{dq}=F'=-F,$$

or

$$\frac{dT_e}{dq}=F;$$

whence, by the reasoning above employed, we arrive at the equation

$$T_e=\frac{L_1 i_1^2}{2}+Mi_1 i_2+\frac{L_2 i_2^2}{2}.$$

Therefore, when there is mechanical motion, we have

$$T=T_e+T_m=\frac{L_1 i_1^2}{2}+Mi_1 i_2+\frac{L_2 i_2^2}{2}+T_m,$$

where T_m is a quadratic function of the $\dot{q}$'s with coefficients functions of the q's, or in Maxwell's notation

$$T=\frac{L_1}{2}\dot{y}_1^2+M\dot{y}_1\dot{y}_2+\frac{L_2}{2}\dot{y}_2^2+T_m.$$

If, then, the system free to move were acted on by any electromotive forces Y_1 and Y_2, we should have

$$\frac{d}{dt}\cdot\frac{dT}{d\dot{y}}-\frac{dT}{dy}=Y_1,\qquad \frac{d}{dt}\cdot\frac{dT}{d\dot{y}_2}-\frac{dT}{dy}=Y_2;$$

or, since $\dfrac{dT}{dy}$ is zero,

$$\frac{d}{dt}(L_1\dot{y}_1+M\dot{y}_2)=Y_1,\qquad (L_2\dot{y}_2+M\dot{y}_1)=Y_2;$$

i.e. in our notation

$$\frac{d}{dt}(L_1 i_1+Mi_2)=E_1,\qquad \frac{d}{dt}(L_2 i_2+Mi_1)=E_2.$$

These results indicate, as previously shewn, the existence of inductive forces in the two circuits equal to $-\dfrac{d}{dt}(Mi_2)$ and $-\dfrac{d}{dt}(Mi_1)$ respectively.

385.] The total draw upon the batteries per unit of time is, as we know, $E_1 i_1+E_2 i_2$.

That is to say, it is

$$\frac{d}{dt}\left(\frac{L_1 i_1^2}{2} + M i_1 i_2 + \frac{L_2 i_2^2}{2}\right) + i_1 i_2 \frac{dM}{dt},$$

or

$$\frac{dT_e}{dt} + \frac{dT_m}{dt},$$

where T_m is the mechanical work done by the electromagnetic force.

If the change in the system were such that the current intensities remained constant, this would become

$$2 i_1 i_2 \frac{dM}{dt}, \quad \text{or} \quad 2\frac{dT_m}{dt}.$$

If we introduce the resistances R_1 and R_2 but continue to neglect T_m, our equations become as before

$$\frac{d}{dt}(L_1 i_1 + M i_2) + R_1 i_1 = E_1,$$

$$\frac{d}{dt}(L_2 i_2 + M i_1) + R_2 i_2 = E_2.$$

386.] Suppose therefore that we have two closed circuits C_1 and C_2 of resistances R_1 and R_2 in any field. Let C_1 be called the primary and C_2 the secondary circuit, and let there be no impressed electromotive force in C_2.

Our equations give

$$E_2 = R_2 i_2 + \frac{d}{dt}(L_2 i_2 + M i_1) = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$R_2 \int i_2 dt + L_2 i_2 + M i_1 = \text{constant};$$

$$\therefore \quad R_2 \int_0^t i_2 dt = (L_2 i_2)_0 + (M i_1)_0 - (L_2 i_2)_1 - (M i_1)_1, \quad . \quad . \quad . \quad (2)$$

where the suffixes 0 and 1 indicate values at the beginning and end of the time t.

From the equation (1) we have, supposing C_2 to be rigid,

$$L_2 \frac{di_2}{dt} + R_2 i_2 = -\frac{d}{dt}(M i_1),$$

or

$$i_2 = e^{-\frac{R_2}{L_2}t} \int \frac{-e^{\frac{R_2}{L_2}t}}{L_2} \frac{d}{dt}(M i_1) dt,$$

supposing i_2 to be zero at the commencement of the time.

If there be any number of closed currents it may be shewn, by precisely the same reasoning as in the case of two currents, that the inductive electromotive force in the circuit of any one of them, as i, is $-\frac{dp}{dt}$, where $p = Li + \Sigma Mi'$ or $\frac{dT_e}{di}$, that is, that the inductive electromotive force is $-\frac{d}{dt}\cdot\frac{dT_e}{di}$.

In the case of two closed currents, i_1 and i_2, it was shewn that $\frac{dT_e}{di_1}$ or $L_1 i_1 + M i_2$ was equal to the flux of magnetic induction through any closed surface S, bounded by the circuit of i_1, that is to say, to the surface integral $\iint \{la + mb + nc\}\, dS$, where a, b, c are the components of magnetic induction of the whole field at any point of S, or to the line integral

$$\int \left\{ F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} \right\} ds,$$

round the circuit of i_1, where F, G, H are the components of vector potential of the whole field at any point of that circuit.

So by reasoning in all respects the same, it may be shewn that whatever be the number of closed currents in the field, $Li + \Sigma Mi'$ or $\frac{dT_e}{di}$ is equal to either of the above expressions, in which a, b, c or F, G, H refer to the whole field.

388.] We proceed now to the investigation of expressions for the electromotive force in a closed circuit, either at rest or in motion, in a varying magnetic field.

First suppose the circuit to be at rest.

We have seen that the total electromotive force in any closed circuit is $-\frac{dp}{dt}$, where

$$p = \int \left\{ F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} \right\} ds,$$

and the integration is taken round the complete circuit.

That is to say, if σ be the total resistance in the complete circuit calculated on the principles enunciated in Chap. XIII, the current i in the circuit will be given by the equation

$$\sigma i = -\int\left\{\frac{dF}{dt}\cdot\frac{dx}{ds}+\frac{dG}{dt}\frac{dy}{ds}+\frac{dH}{dt}\frac{dz}{ds}\right\}ds. \quad . \quad . \quad . \quad \text{(A)}$$

If at each point of the circuit there were an electromotive force with components P, Q, R, and if σ_1 were the resistance per unit length in the circuit at that point, we should have

$$\sigma_1 i = P\frac{dx}{ds}+Q\frac{dy}{ds}+R\frac{dz}{ds},$$

i being the same at each point of the circuit and equal to the value given by equation (A).

The magnitude σ_1 varies generally from point to point of the circuit, hence we infer that

$$P=-\left(\frac{dF}{dt}+P'\right),\quad Q=-\left(\frac{dG}{dt}+Q'\right),\quad R=-\left(\frac{dH}{dt}+R'\right);$$

where $$\int\left(P'\frac{dx}{ds}+Q'\frac{dy}{ds}+R'\frac{dz}{ds}\right)ds=0.$$

From this equation it follows that P', Q', R' are derivatives of some function ψ, which satisfies the condition that

$$\frac{\dfrac{dF}{dt}\cdot\dfrac{dx}{ds}+\dfrac{dG}{dt}\cdot\dfrac{dy}{ds}+\dfrac{dH}{dt}\cdot\dfrac{dz}{ds}+\dfrac{d\psi}{ds}}{\sigma_1},$$

is constant throughout the circuit, and therefore that the most general forms of the expressions for the components of the electromotive force at any point of the circuit are given by the equations

$$P=-\frac{dF}{dt}-\frac{d\psi}{dx},\quad Q=-\frac{dG}{dt}-\frac{d\psi}{dy},\quad R=-\frac{dH}{dt}-\frac{d\psi}{dz}.$$

389]. In the next place, suppose that the circuit is not at rest, but is varying in form, or position, or both, from time to time.

Consider any element ds of this changing circuit. The electromotive force of induction in ds may be regarded as the sum of two parts which may be separately calculated, viz.

(1) that arising from the intrinsic variation of the surrounding field and which would exist if the element ds were at rest;

(2) that arising from the motion of ds.

The first has for its components the magnitudes

$$-\frac{dF}{dt},\quad -\frac{dG}{dt},\quad -\frac{dH}{dt},$$

or more generally,

$$-\frac{dF}{dt}-\frac{d\psi}{dx}, \quad -\frac{dG}{dt}-\frac{d\psi}{dy}, \quad -\frac{dH}{dt}-\frac{d\psi}{dz},$$

where $\frac{dF}{dt}$, $\frac{dG}{dt}$, and $\frac{dH}{dt}$ are the time variations of the F, G, H of the whole surrounding field (including the remainder of the circuit) at ds.

390.] To calculate the second, we observe that it must be the same as if ds were itself at rest, and the remaining field unchanged in all other respects, moved in a space animated with the reversed motion of ds.

Hence, if $\dot{x}, \dot{y}, \dot{z}$ were the component translational velocities of ds, and ω_1, ω_2, ω_3 its component rotational velocities, the time variations of F, G, and H, referred to axes fixed relatively to the element and instantaneously coinciding with those of reference, would be, for F,

$$\frac{dF}{dx}\dot{x}+\frac{dF}{dy}\dot{y}+\frac{dF}{dz}\dot{z}+\omega_3 G-\omega_2 H,$$

with similar expressions for G and H, since F, G, H are components of a vector.

Therefore the components P, Q, R of the electromotive force in ds arising both from the variation of the field and the motion of ds are given by the equations,

$$-P=\dot{x}\frac{dF}{dx}+\dot{y}\frac{dF}{dy}+\dot{z}\frac{dF}{dz}+\omega_3 G-\omega_2 H+\frac{dF}{dt}+\frac{d\psi}{dx},$$

with similar equations for Q and R.

Since $\quad \frac{dF}{dy}-\frac{dG}{dx}=-c \quad$ and $\quad \frac{dF}{dz}-\frac{dH}{dx}=b,\quad$ we get

$$-P=\dot{x}\frac{dF}{dx}+\dot{y}\frac{dG}{dx}+\dot{z}\frac{dH}{dx}-c\dot{y}+b\dot{z}+\omega_3 G-\omega_2 H+\frac{dF}{dt}+\frac{d\psi}{dx},$$

$$-Q=\dot{x}\frac{dF}{dy}+\dot{y}\frac{dG}{dy}+\dot{z}\frac{dH}{dy}-a\dot{z}+c\dot{x}+\omega_1 H-\omega_3 F+\frac{dG}{dt}+\frac{d\psi}{dy},$$

$$-R=\dot{x}\frac{dF}{dz}+\dot{y}\frac{dG}{dz}+\dot{z}\frac{dH}{dz}-b\dot{x}+a\dot{y}+\omega_2 F-\omega_1 G+\frac{dH}{dt}+\frac{d\psi}{dz}.$$

391.] The electromotive force round the circuit, or

$$\int\left(P\frac{dx}{ds}+Q\frac{dy}{ds}+R\frac{dz}{ds}\right)ds,$$

becomes in this case, since the continuity of the circuit requires that

$$\frac{d\dot{x}}{ds} = \omega_2 \frac{dz}{ds} - \omega_3 \frac{dy}{ds}, \quad \frac{d\dot{y}}{ds} = \omega_3 \frac{dx}{ds} - \omega_1 \frac{dz}{ds}, \quad \frac{d\dot{z}}{ds} = \omega_1 \frac{dy}{ds} - \omega_2 \frac{dx}{ds},$$

$$\int \left\{ \left((c\dot{y} - b\dot{z}) - \frac{dF}{dt}\right)\frac{dx}{ds} + \left((a\dot{z} - c\dot{x}) - \frac{dG}{dt}\right)\frac{dy}{ds} + \left((b\dot{x} - a\dot{y}) - \frac{dH}{dt}\right)\frac{dz}{ds} \right\} ds$$
$$- \int \frac{d}{ds}(F\dot{x} + G\dot{y} + H\dot{z})\, ds.$$

And the part under the integral sign in the last term being a complete differential the term itself vanishes.

For closed circuits, therefore, from which alone our experimental evidence is derived, it is indifferent whether we take for P, Q, R the values obtained above, or those given by the simpler equations,

$$P = c\dot{y} - b\dot{z} - \frac{dF}{dt} - \frac{d\psi}{dx}, \quad Q = a\dot{z} - c\dot{x} - \frac{dG}{dt} - \frac{d\psi}{dy},$$
$$R = b\dot{x} - a\dot{y} - \frac{dH}{dt} - \frac{d\psi}{dz};$$

and these simpler expressions are most usually adopted.

392.] We proceed now to determine the electromagnetic action on a closed circuit or any element of such a circuit in any magnetic field.

By Art. 326 we know that, if a closed circuit with current of intensity i be situated in a magnetic field, the total increase of material kinetic energy corresponding to any variation in the form and position of the circuit is equal to the corresponding variation in the integral

$$\int i\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds;$$

that is to say to

$$\int i \left\{ \left(\frac{dF}{dx}\frac{dx}{ds} + \frac{dG}{dx}\frac{dy}{ds} + \frac{dH}{dx}\frac{dz}{ds}\right)\delta x + \left(\frac{dF}{dy}\frac{dx}{ds} + \frac{dG}{dy}\frac{dy}{ds} + \frac{dH}{dy}\frac{dz}{ds}\right)\delta y \right.$$
$$\left. + \left(\frac{dF}{dz}\frac{dx}{ds} + \frac{dG}{dz}\frac{dy}{ds} + \frac{dH}{dz}\frac{dz}{ds}\right)\delta z \right\} ds$$
$$+ \int i \left\{ F\delta\frac{dx}{ds} + G\delta\frac{dy}{ds} + H\delta\frac{dz}{ds} \right\} ds.$$

The coefficient of δx under the above integral sign may be written

$$\frac{dF}{dx}\frac{dx}{ds}+\frac{dF}{dy}\frac{dy}{ds}+\frac{dF}{dz}\frac{dz}{ds}+\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right)$$

or
$$\frac{dF}{ds}+\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right),$$

with similar modifications in the coefficients of dy and dz.

Therefore the total variation of the integral becomes, since

$$\delta\frac{dx}{ds}=\frac{d}{ds}\delta x,\quad \delta\frac{dy}{ds}=\frac{d}{ds}\delta y,\quad \delta\frac{dz}{ds}=\frac{d}{ds}\delta z,$$

$$\int i\left\{\frac{d}{ds}(F\delta x+G\delta y+H\delta z)+\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right)\delta x+\left(a\frac{dz}{ds}-c\frac{dx}{ds}\right)\delta y\right.$$
$$\left.+\left(b\frac{dx}{ds}-a\frac{dy}{ds}\right)\delta z\right\}ds.$$

And this for the closed circuit is equivalent to

$$\int i\left\{\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right)\delta x+\left(a\frac{dz}{ds}-c\frac{dx}{ds}\right)\delta y+\left(b\frac{dx}{ds}-a\frac{dy}{ds}\right)\delta z\right\}ds;$$

or the total increase of the material kinetic energy for any change in form and position of the circuit is the same as if each element ds were acted on by the force whose components are

$$i\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right)ds,\quad i\left(a\frac{dz}{ds}-c\frac{dx}{ds}\right)ds,\quad i\left(b\frac{dx}{dy}-a\frac{dy}{ds}\right)ds.$$

If we include the term

$$\frac{d}{ds}(F\delta x+G\delta y+H\delta z),$$

remembering that from the continuity of the circuit

$$\delta\left(\frac{dx}{ds}\right)=\delta\phi\frac{dz}{ds}-\delta\psi\frac{dy}{ds},\qquad \delta\frac{dy}{ds}=\delta\psi\frac{dx}{ds}-\delta\theta\frac{dz}{dx},$$
$$\delta\frac{dz}{ds}=\delta\theta\frac{dy}{ds}-\delta\phi\frac{dx}{ds},$$

where $\delta\theta$, $\delta\phi$, $\delta\psi$ are angular displacements of the element ds round the axes, the total increase of the kinetic energy may be written

$$\int i\left\{\left(\frac{dF}{ds}+c\frac{dy}{ds}-b\frac{dz}{ds}\right)\delta x+\left(\frac{dG}{ds}+a\frac{dz}{ds}-c\frac{dx}{ds}\right)\delta y\right.$$
$$\left.+\left(\frac{dH}{ds}+b\frac{dx}{ds}-a\frac{dy}{ds}\right)\delta z\right\}ds$$
$$+\int i\left\{\left(H\frac{dy}{ds}-G\frac{dz}{ds}\right)\delta\theta+\left(F\frac{dz}{ds}-H\frac{dx}{ds}\right)\delta\phi\right.$$
$$\left.+\left(G\frac{dx}{ds}-F\frac{dy}{ds}\right)\delta\psi\right\}ds;$$

that is to say it is the same as if each element ds were acted on by the last mentioned force together with another whose components are

$$i\frac{dF}{ds}\,ds,\quad i\frac{dG}{ds}\,ds,\quad i\frac{dH}{ds}\,ds,$$

and also a moment whose components are

$$i\left(H\frac{dy}{ds}-G\frac{dz}{ds}\right)ds,\quad i\left(F\frac{dz}{ds}-H\frac{dx}{ds}\right)ds,\quad i\left(G\frac{dx}{ds}-F\frac{dy}{ds}\right)ds.$$

Since

$$\int i\frac{dF}{dx}\,ds,\qquad \int i\frac{dG}{ds}\,ds,\qquad \int i\frac{dH}{ds}\,ds$$

separately vanish for a closed circuit, the component forces on the whole circuit are the same whichever view be adopted.

Also, since the total moment round the axis of z on the second hypothesis is equal to

$$\int i\left\{y\frac{dF}{ds}-x\frac{dG}{ds}+F\frac{dy}{ds}-G\frac{dx}{ds}+y\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right)\right.$$
$$\left.-x\left(a\frac{dz}{ds}-c\frac{dx}{ds}\right)\right\}ds$$

$$\text{i.e. to}\quad \int i\left\{y\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right)-x\left(a\frac{dz}{ds}-c\frac{dx}{ds}\right)\right\}ds,$$

it follows that this is equal to the total moment round the same axis of the forces existing on the first hypothesis, and similarly for the moments round x and y

393.] It appears therefore that in dealing with a complete rigid circuit we may take indifferently for the action on each element either the component forces obtained in the second instance with the corresponding moments, or the component forces given by the simpler expressions

$$i\left(c\frac{dy}{ds}-b\frac{dz}{ds}\right)\delta s,\quad i\left(a\frac{dz}{ds}-c\frac{dx}{ds}\right)\delta s,\quad i\left(b\frac{dx}{ds}-a\frac{dy}{ds}\right)\delta s,$$

with moments zero.

So long as experiments are conducted with rigid circuits it is impossible to decide between the two results, but certain experiments on flexible circuits have been interpreted as pointing to the conclusion that the electromagnetic action on each element is a single force perpendicular to the element, and therefore that it is given by the component forces last written.

394.] In these investigations we have supposed that we were dealing with circuits in linear conductors (like copper wires) either rigid or flexible, and have determined the electromotive forces and electromagnetic actions on each element from the condition that such forces and actions shall be consistent with certain known experimental results when the whole circuit is considered.

If, instead of a single linear circuit, or an aggregate of detached linear circuits, we had to deal with a continuous conducting mass, we should infer that at each point of such a mass there is an electromotive force whose components are the P, Q, R determined by the preceding investigation, and therefore that elementary currents will be set up within the substance, such that if u, v, w were their components and σ the resistance referred to unit of area at any point,

$$\sigma u = P,\qquad \sigma v = Q,\qquad \sigma w = R.$$

If u, v, w satisfy the condition

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$$

at each point, we may divide the whole mass into closed circuits of appropriate transverse section a from point to point, and such that Ia is the same throughout each circuit, where

$$I=\sqrt{u^2+v^2+w^2}.$$

But Ia corresponds to the i of the linear circuits hitherto considered, and

$$i\frac{dx}{ds}ds = I\frac{dx}{ds}a\,ds = u\,dx\,dy\,dz,\ \&c.$$

$$F = \iint \frac{u_s}{r} dS = \iint \frac{1}{r}\left(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS,$$
$$G \qquad = \iint \frac{1}{r}\left(l\frac{d\phi}{dz} - n\frac{d\phi}{dx}\right) dS,$$
$$H \qquad = \iint \frac{1}{r}\left(m\frac{d\phi}{dx} - l\frac{d\phi}{dy}\right) dS.$$

If the surface S be closed, we have by Art. 271,

$$\iint\left(n\frac{d}{dy} - m\frac{d}{dz}\right)\frac{\phi}{r} dS = 0, \text{ \&c.}$$

And therefore

$$F = \iint \phi\left(m\frac{d}{dz} - n\frac{d}{dy}\right)\frac{1}{r} dS,$$
$$G = \iint \phi\left(n\frac{d}{dx} - l\frac{d}{dz}\right)\frac{1}{r} dS,$$
$$H = \iint \phi\left(l\frac{d}{dy} - m\frac{d}{dx}\right)\frac{1}{r} dS;$$

so that F, G, H are linear function of the ϕ's with coefficient functions of the coordinates. Evidently the same is true of the derived functions $\frac{dF}{dx}$, $\frac{dF}{dy}$, &c.

COROLLARY. The vector potential due to any spherical current sheet is tangential to any spherical surface concentric with the sheet. For, taking the centre for origin, let x, y, z refer to a point on the sheet, x', y', z' to a point on the concentric surface; and let

$$D^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

Then $$F = \iint \phi \frac{zy' - yz'}{D^3} dS, \qquad G = \iint \phi \frac{xz' - zx'}{D^3} dS,$$
$$H = \iint \phi \frac{yx' - xy'}{D^3} dS;$$

and therefore $x'F + y'G + z'H = 0$, which proves the statement.

The Energy of a System of Current Sheets.

411.] The energy of any system of current sheets can be put in the form

$$2T = \iiint (Fu_s + Gv_s + Hw_s)\, dx\, dy\, dz$$

extended over all the currents. But for every current sheet, u_s, v_s, w_s are subject to the condition $lu_s + mv_s + nw_s = 0$. The

expression contains therefore more variables than it has degrees of freedom, and it becomes desirable to transform it, by substituting ϕ, the current function, as the independent variable.

Given any system of current sheets, let us apply the theorem of Art. 271, using the function $F\phi$ for P of that article. That gives

$$\int F\phi \frac{dx}{ds}\, ds = \iint \Big(m\frac{d}{dz} - n\frac{d}{dy}\Big)(F\phi)\, dS,$$

the first integral being round the bounding curve of each surface and the second over all the surfaces. That is,

$$\int F\phi \frac{dx}{ds}\, ds = \iint F\Big(m\frac{d\phi}{dz} - n\frac{d\phi}{dy}\Big)\, dS + \iint \phi\Big(m\frac{dF}{dz} - n\frac{dF}{dy}\Big) dS.$$

Treating $G\phi$ and $H\phi$ in the corresponding way, we obtain

$$\iint \Big\{ F\Big(n\frac{d\phi}{dy} - m\frac{d\phi}{dz}\Big) + G\Big(l\frac{d\phi}{dz} - n\frac{d\phi}{dx}\Big) + H\Big(m\frac{d\phi}{dx} - l\frac{d\phi}{dy}\Big)\Big\}\, dS$$
$$= \iint \phi\Big\{ l\Big(\frac{dH}{dy} - \frac{dG}{dz}\Big) + m\Big(\frac{dF}{dz} - \frac{dH}{dx}\Big) + n\Big(\frac{dG}{dx} - \frac{dF}{dy}\Big)\Big\}\, dS$$
$$- \int \phi\Big(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\Big)\, ds.$$

The first member is equal to $2T$.

We will now suppose all the surfaces closed. Then the second term of the right-hand member vanishes, and therefore for any system of closed current sheets,

$$2T = \iint \phi\Big\{ l\Big(\frac{dH}{dy} - \frac{dG}{dz}\Big) + m\Big(\frac{dF}{dz} - \frac{dH}{dx}\Big) + n\Big(\frac{dG}{dx} - \frac{dF}{dy}\Big)\Big\}\, dS.$$

Also if Ω be the magnetic potential and $\mu = 1$,

$$\frac{dG}{dz} - \frac{dH}{dy} = \frac{d\Omega}{dx}, \quad \frac{dH}{dx} - \frac{dF}{dz} = \frac{d\Omega}{dy}, \quad \text{and} \quad \frac{dF}{dy} - \frac{dG}{dx} = \frac{d\Omega}{dz}.$$

Therefore

$$2T = -\iint \phi\Big(l\frac{d\Omega}{dx} + m\frac{d\Omega}{dy} + n\frac{d\Omega}{dz}\Big)\, dS$$
$$= -\iint \phi\frac{d\Omega}{d\nu}\, dS.$$

412.] It is necessary now to show that $\dfrac{d\Omega}{d\nu}$ is continuous through any current sheet. For this purpose it is sufficient to take the tangent plane at any point for the plane of x, y. We then have

$$\frac{d\Omega}{d\nu} = \frac{d\Omega}{dz} = \frac{dF}{dy} - \frac{dG}{dx}.$$

b_1' and c_1'. These conditions together with equations (I) are satisfied by the assumption

$$F_1 = \mu \iiint \frac{u}{r}\,dx\,dy\,dz, \qquad G_1 = \mu \iiint \frac{v}{r}\,dx\,dy\,dz,$$

$$H_1 = \mu \iiint \frac{w}{r}\,dx\,dy\,dz.$$

The difference of the magnetic forces (a_2, β_2, γ_2) and $(a_2', \beta_2', \gamma_2')$ in the direction of the normal on opposite sides of the surface is

$$4\pi \{l(A-A')+m(B-B')+n(C-C')\},$$

where $\quad 4\pi A = \frac{\mu-1}{\mu}a \quad$ and $\quad 4\pi A' = \frac{\mu'-1}{\mu'}a'$, &c., &c.

Therefore

$$a_2' = a_2 + \frac{\mu-1}{\mu}l\,(la+mb+nc) - \frac{\mu'-1}{\mu'}l\,(la'+mb'+nc'),$$

$$\frac{a_2'}{\mu'} = \frac{a_2}{\mu} + \ldots \text{ &c.}$$

Also $\frac{a_1'}{\mu'} = \frac{a_1}{\mu}$;

$$\therefore \quad \frac{a'}{\mu'} = \frac{a}{\mu} + l\left\{\frac{\mu-1}{\mu}(la+mb+nc) - \frac{\mu'-1}{\mu'}(la'+mb'+nc')\right\}. \quad \text{(II)}$$

Hence F, G, H at any point of the field are to be determined from the equations

$$F = F_1 + F_2, \qquad G = G_1 + G_2, \qquad H = H_1 + H_2,$$

where F_1, G_1, H_1 are determined as above, and F_2, G_2, H_2 are determined from the equations

$$\nabla^2 F_2 = \nabla^2 G_2 = \nabla^2 H_2 = 0,$$

together with the three superficial equations corresponding to (II).

399.] The total energy E in any field of electric currents in which there is no material motion consists of three parts:

(1) The electrokinetic energy T_e which is equal, as we have shown, to

$$\frac{1}{2}\iiint \{Fu + Gv + Hw\}\,dx\,dy\,dz.$$

(2) The dissipated energy or heat D, which is equal to

$$\iiint \left\{\int \sigma(p^2+q^2+r^2)\,dt\right\} dx\,dy\,dz.$$

(3) The potential energy of electrical distribution W, which, supposing the field to be isotropic, is equal to

$$\frac{2\pi}{K}\cdot\iiint\{f^2+g^2+h^2\}\,dx\,dy\,dz.$$

Now since $F=\mu\iiint\frac{u\,dx\,dy\,dz}{r}$, it follows that

$$u\frac{dF}{du}=F,$$

and that

$$u\frac{dF}{dt}=F\frac{du}{dt}.$$

Therefore

$$\begin{aligned}\frac{dT_e}{dt}&=\iiint\left\{u\frac{dF}{dt}+v\frac{dG}{dt}+w\frac{dH}{dt}\right\}dx\,dy\,dz\\&=-\iiint\{P(p+\dot f)+Q(q+\dot g)+R(r+\dot h)\}\,dx\,dy\,dz\\&\quad-\iiint\left\{u\frac{d\psi}{dx}+v\frac{d\psi}{dy}+w\frac{d\psi}{dz}\right\}dx\,dy\,dz.\end{aligned}$$

Also

$$\begin{aligned}\frac{dD}{dt}&=\iiint\sigma(p^2+q^2+r^2)\,dx\,dy\,dz\\&=\iiint(Pp+Qq+Rr)\,dx\,dy\,dz.\end{aligned}$$

Since $\sigma p=P,\quad \sigma q=Q,\quad \sigma r=R.$

And

$$\begin{aligned}\frac{dW}{dt}&=\frac{4\pi}{K}\iiint\{f\dot f+g\dot g+h\dot h\}\,dx\,dy\,dz,\\&=\iiint\{P\dot f+Q\dot g+R\dot h\}\,dx\,dy\,dz.\end{aligned}$$

Since $f=\frac{K}{4\pi}P,\quad g=\frac{K}{4\pi}Q,\quad h=\frac{K}{4\pi}R.$

Whence, by addition, we get

$$\begin{aligned}\frac{dE}{dt}=\frac{dT_e}{dt}+\frac{dD}{dt}+\frac{dW}{dt}&=-\iiint\left\{u\frac{d\psi}{dx}+v\frac{d\psi}{dy}+w\frac{d\psi}{dz}\right\}dx\,dy\,dz\\&=-\iint\psi(lu+mv+nw)\,dS+\iiint\psi\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)dx\,dy\,dz,\end{aligned}$$

where S is a surface bounding the whole field.

But the first of these terms is clearly zero, and the second is so also provided the currents be all closed, in virtue of the equation

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

Therefore $$\frac{dE}{dt} = \frac{dT_e}{dt} + \frac{dD}{dt} + \frac{dW}{dt} = 0.$$

400.] If there be material motion in the field, then we know that the time variation $\frac{dT_e}{dt}$ must be increased by the quantity

$$\iiint \{P'u + Q'v + R'w\}\, dx\, dy\, dz,$$

where P', Q', R' are the additions to be made to P, Q, R arising from the motion of the element.

Now if we are dealing with a field of closed circuits, we know that

$$\left(\iiint \{P'u + Q'v + R'w\}\, dx\, dy\, dz\right)$$

is equal to $$-\Sigma \int i\left(\frac{dF}{dt}\cdot\frac{dx}{ds} + \frac{dG}{dt}\cdot\frac{dy}{ds} + \frac{dH}{dt}\cdot\frac{dz}{ds}\right) ds,$$

where the line integral refers to any one of the closed circuits.

Also if q be a generalised coordinate of form or position of circuit, $\frac{dF}{dt} = \Sigma \frac{dF}{dq}\cdot\frac{dq}{dt}$, and similarly for $\frac{dG}{dt}$ and $\frac{dH}{dt}$.

But in this case there is, as above proved, material kinetic energy T_m, such that

$$\frac{dT_m}{dt} = \Sigma \int i\left(\frac{dF}{dt}\cdot\frac{dx}{ds} + \frac{dG}{dt}\cdot\frac{dy}{ds} + \frac{dH}{dt}\cdot\frac{dz}{ds}\right) ds,$$

where $\frac{dF}{dt}$, $\frac{dG}{dt}$, $\frac{dH}{dt}$ have the last mentioned values.

Whence it follows that

$$\iiint \{P'u + Q'v + R'w\}\, dx\, dy\, dz + \frac{dT_m}{dt} = 0.$$

And we get as before

$$\frac{dE}{dt} = \frac{dT_e}{dt} + \frac{dD}{dt} + \frac{dW}{dt} + \frac{dT_m}{dt} = 0.*$$

* As shown above, Art. 391,

$\iiint \{P'u + Q'v + R'w\}\, dx\, dy\, dz$ in a field of closed circuits

$$= \iiint \{(c\dot{y} - b\dot{z})u + (a\dot{z} - c\dot{x})v + (b\dot{x} - a\dot{y})w\}\, dx\, dy\, dz.$$

Also, by Art. 392,

$$\frac{dT_m}{dt} = \iiint \{X\dot{x} + Y\dot{y} + Z\dot{z}\}\, dx\, dy\, dz$$

$$= \iiint \{(cv - bw)\dot{x} + (aw - cu)\dot{y} + (bu - av)\dot{z}\}\, dx\, dy\, dz,$$

whence the result as in the text.

401.] The expression T_e for the electrokinetic energy, or

$$\frac{1}{2}\iiint\{Fu+Gv+Hw\}\,dx\,dy\,dz,$$

differs, as shown above (Art. 357), from the integral

$$\frac{1}{8\pi\mu}\cdot\iiint\{a^2+b^2+c^2\}\,dx\,dy\,dz,$$

only by a surface integral over a surface infinitely distant, and therefore either volume integral may be taken to express the electrokinetic energy over the whole of space.

If, with Maxwell, we regard this electrokinetic energy as localised in regions of magnetic force (a, b, c) rather than those of currents (u, v, w), then in dealing with a finite space we shall take the electrokinetic energy to be

$$\frac{1}{8\pi\mu}\iiint\{a^2+b^2+c^2\}\,dx\,dy\,dz$$

throughout this space, and the time variation of this integral will be the time variation of the electrokinetic energy T_e within the space.

The dissipated energy or heat we regard as expressed locally by the equation

$$D=\iiint\left\{\int\sigma\,dt(p^2+q^2+r^2)\right\}dx\,dy\,dz,$$

where the integral is taken as before over the given space.

Similarly, the potential energy or W we regard as expressed locally by the equation

$$W=\frac{2\pi}{K}\iiint(f^2+g^2+h^2)\,dx\,dy\,dz.$$

With these assumptions it may be proved that the total variation per unit of time of E, or

$$T_e+D+W+T_m,$$

within the space bounded by a closed surface S is equal to the surface integral

$$\frac{1}{4\pi\mu}\cdot\iint\{l(Rb-Qc)+m(Pc-Ra)+n(Qa-Pb)\}\,dS,$$

where P, Q, R are the components of the electromotive force at

$$\left.\begin{aligned}\frac{d}{dt}\frac{dT}{du}&=0\\ \frac{d}{dt}\frac{dT}{dv}&=0\\ \frac{d}{dt}\frac{dT}{dw}&=0\end{aligned}\right\}\text{ at all points on } S,$$

with the condition that induced currents can exist only in the shell.

If we take the common centre of the spheres for origin, this condition is

$$\frac{x}{a}u+\frac{y}{a}v+\frac{z}{a}w=0,$$

at each point in the shell.

It will be found in this case that if we determine u, v, w by using the equations $\frac{d}{dt}\frac{dT}{du}=0$, &c., without regard to the condition, the values so found, in fact, satisfy the condition, and correspond to a system of closed superficial currents on S. They are therefore the solution of the problem.

Since $\frac{dT}{du}=F$, &c., by making $\frac{d}{dt}\frac{dT}{du}=0$, &c., unconditionally, we obtain $\frac{dF_0}{dt}+\frac{dF}{dt}=0$, &c., and therefore since the motion is from rest,

$$F_0+F=0, \qquad G_0+G=0, \qquad H_0+H=0,$$

at all points on S.

Then the currents u_0, v_0, w_0 have a current function ϕ, which can be expressed in a series of spherical harmonics referred to the common centre of the spheres. It is sufficient to treat one harmonic term in this expression. Let therefore ϕ be a solid harmonic of order n. Then, taking the centre for origin, u_0, v_0, w_0 are the values on S_0 of the functions,

$$u_0=\frac{z}{a_0}\frac{d\phi}{dy}-\frac{y}{a_0}\frac{d\phi}{dz},$$

$$v_0=\frac{x}{a_0}\frac{d\phi}{dz}-\frac{z}{a_0}\frac{d\phi}{dx},$$

$$w_0=\frac{y}{a_0}\frac{d\phi}{dx}-\frac{x}{a_0}\frac{d\phi}{dy}.$$

And therefore u_0, v_0, w_0 are spherical harmonics of order n.

Suppose on S_0, $u_0 = B_n Y_n$; then on S, $F_0 = \frac{4\pi a}{2n+1}\left(\frac{a}{a_0}\right)^n B_n Y_n$, by Art. 66.

And therefore
$$F = -\frac{4\pi a}{2n+1}\left(\frac{a}{a_0}\right)^n B_n Y_n,$$
and therefore on S
$$u = -\left(\frac{a}{a_0}\right)^n B_n Y_n$$
$$= -\left(\frac{a}{a_0}\right)^n u_0.$$

Similarly
$$v = -\left(\frac{a}{a_0}\right)^n v_0, \qquad w = -\left(\frac{a}{a_0}\right)^n w_0.$$

Hence we see that u, v, w are symmetrical with respect to u_0, v_0, w_0; and since u_0, v_0, w_0 constitute a system of closed currents on the outer sphere, u, v, w constitute a system of closed currents on S.

418.] In the general case, the equations $\frac{dF_0}{dt} + \frac{dF}{dt} = 0$, &c., will determine a system of values for u, v, w on S, which do not satisfy the condition $lu + mv + nw = 0$. We must therefore have recourse to another method of solution *.

A General Solution.

Let there be certain surfaces $S_1, S_2, \dots S_r$ on which ϕ, the current function, is given as a function of the time at each point, constituting a given varying magnetic field; and certain

* It may be suggested that we should apply Lagrange's equation to the expression for energy,
$$2T = \iiint (Fu + Gv + Hw)\, dx\, dy\, dz,$$
having regard to the condition $lu + mv + nw = 0$. We should thus obtain
$$\frac{dF}{dt} = l\frac{d\lambda}{dt}, \quad \frac{dG}{dt} = m\frac{d\lambda}{dt}, \quad \frac{dH}{dt} = n\frac{d\lambda}{dt},$$
when $\frac{d\lambda}{dt}$ is an indeterminate multiplier.

Here λ is evidently the resultant vector potential on S; and the equations show it to be normal to S at every point. By this method we should obtain determinate values for u, v, w on S, satisfying the condition $lu + mv + nw = 0$. But, as will be shown, they cannot satisfy the condition of continuity except in the case where $\lambda = 0$.

remaining surfaces $S_{r+1}, \dots S_n$, on which ϕ is to be determined by induction.

Now there are no forces tending to increase ϕ on $S_{r+1}, \dots S_n$ except the forces of induction. The potential of free electricity, if it exist, and the forces derived from it, can have no effect on ϕ. Further, the system has as many degrees of freedom as it contains variables, namely the ϕ's.

If therefore the external magnetic system be generated continuously, the corresponding system of induced currents on $S_{r+1}, \dots S_n$ will be determined by the equations

$$\frac{d}{dt}\frac{dT}{d\phi} = 0$$

at each point on the surfaces $S_{r+1}, \dots S_n$.

That is,
$$\frac{d}{dt}\left(\frac{d\Omega_0}{dv} + \frac{d\Omega}{dv}\right) = 0.$$

That is,
$$\frac{d}{dv}\left(\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt}\right) = 0$$

at each point on $S_{r+1}, \dots S_n$;

also
$$\nabla^2\frac{d\Omega_0}{dt} = 0;$$

and
$$\nabla^2\frac{d\Omega}{dt} = 0;$$

at all points within any of those surfaces; and therefore $\dfrac{d\Omega_0}{dt} + \dfrac{d\Omega}{dt}$ has uniform value over and within each of the surfaces $S_{r+1}, \dots S_n$.

But, as already proved, there is for each surface only one determinate system of closed currents which has this property. We see then that the system of closed currents which will be induced on the closed surfaces $S_{r+1}, \dots S_n$, is the determinate system which we found above, making the magnetic potential constant upon or within each of the surfaces $S_{r+1}, \dots S_n$. As any closed currents come into existence outside of $S_{r+1}, \dots S_n$ by the variation of the external system, their magnetic screen is formed on $S_{r+1}, \dots S_n$.

These induced currents decay by resistance, and cease to be

a complete screen. We are here considering only the law of their formation.

419.] On this hypothesis, and neglecting for the present the resistance, the magnetic force will be zero at every point within any of the surfaces $S_{r+1}, \dots S_n$; it remains to consider the electromotive forces.

Since the magnetic force is zero at every point, we have, writing F, G, H for the components of the complete vector potential, due as well to the original as to the induced systems,

$$\frac{d}{dy}\frac{dF}{dt} = \frac{d}{dx}\frac{dG}{dt}, \quad \frac{d}{dx}\frac{dH}{dt} = \frac{d}{dz}\frac{dF}{dt}, \quad \frac{d}{dz}\frac{dG}{dt} = \frac{d}{dy}\frac{dH}{dt},$$

at all points within the surfaces $S_{r+1}, \dots S_n$. It follows that there exists a function ψ, of x, y, and z, such that

$$-\frac{dF}{dt} = \frac{d\psi}{dx}, \quad -\frac{dG}{dt} = \frac{d\psi}{dy}, \quad -\frac{dH}{dt} = \frac{d\psi}{dz},$$

and

$$\nabla^2\psi = -\frac{d}{dt}\left(\frac{dF}{dx} + \frac{dG}{dx} + \frac{dH}{dz}\right) = 0,$$

at every point within any of the surfaces $S_{r+1}, \dots S_n$.

Now $-\frac{dF}{dt}$, $-\frac{dG}{dt}$, and $-\frac{dH}{dt}$ being the components of an electromotive force, produce, according to the theory of electrostatics, on the surface of the conductor S a distribution of electricity having potential ψ, and therefore causing at all points within S an electromotive force equal and opposite to the resultant of $-\frac{dF}{dt}$, $-\frac{dG}{dt}$, and $-\frac{dH}{dt}$. This distribution, and its potential ψ, will be invariable with the time as long as $-\frac{dF}{dt}$, &c., are invariable.

We are thus led to expect that, in response to the variation of the magnetic field outside of a conductor, there will be induced on the conductor (1) a system of electric currents reducing to zero the magnetic force, and (2) a distribution of free electricity on the surface reducing to zero the electromotive force, at all points within the conductor.

420.] The potential function ψ at which we arrived in the last article requires further investigation.

Let S be any closed surface, P, Q, R the components of a vector which satisfy the condition

$$\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=0$$

at all points within S. Then

$$\iint(lP+mQ+nR)\,dS=0.$$

It follows that there exists a function χ of x, y, z, such that

$$\frac{d\chi}{dv}=lP+mQ+nR$$

at all points on S, and $\nabla^2\chi = 0$ at all points within S.

We will call this function the *associated function* to P, Q, R for the surface S.

The vectors
$$P-\frac{d\chi}{dx},\quad Q-\frac{d\chi}{dy},\quad R-\frac{d\chi}{dz}$$
have a resultant in the tangent plane at each point on S.

421.] If P, Q, and R satisfy the further condition

$$\frac{dP}{dy}-\frac{dQ}{dx}=0, \text{ \&c.,}$$

at all points within S, then shall

$$P-\frac{d\chi}{dx}=0,\quad Q-\frac{d\chi}{dy}=0,\quad \text{and}\quad R-\frac{d\chi}{dz}=0$$

at all points within S. For these further conditions being satisfied, there must exist a function χ', such that at all points within S

$$P=\frac{d\chi'}{dx},\quad Q=\frac{d\chi'}{dy},\quad R=\frac{d\chi'}{dz}.$$

And therefore
$$P-\frac{d\chi}{dx}=\frac{d}{dx}(\chi'-\chi), \text{ \&c.}$$

Then $\dfrac{d(\chi'-\chi)}{dv}=0$ at all points on S, and $\nabla^2(\chi'-\chi)=0$ at all points within S. And therefore $\chi'-\chi=$ constant, and $P-\dfrac{d\chi}{dx}=0$ &c. at all points within S.

422.] If P, Q, and R be the components of an electromotive force, and if S be a conducting shell, then, whether the conditions $\dfrac{dP}{dy}=\dfrac{dQ}{dx}$, &c., be satisfied or not, they will maintain on S the

in which σ denotes the specific resistance, be satisfied at every point by some value or other of $\frac{dF}{dt}$, $\frac{dG}{dt}$, $\frac{dH}{dt}$, and Ψ.

Let us suppose the shell to be of uniform material and σ constant. Then we have by differentiation

$$\sigma\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{d}{dt}\left(\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right)+\nabla^2\Psi=0$$

at all points within the substance of the shell. If therefore the currents are derived from a current function, and if F, G, and H relate to any varying magnetic system, including the currents in the shell themselves, it follows that $\nabla^2\Psi=0$ at all points within the substance of the shell.

But also multiplying equations (A) in order by l, m, n, and adding, we have

$$l\frac{dF}{dt}+m\frac{dG}{dt}+n\frac{dH}{dt}+\frac{d\Psi}{d\nu}=-\sigma(lu+mv+nw)$$
$$=0$$

and therefore Ψ is by definition the associated function to $-\frac{dF}{dt}$, $-\frac{dG}{dt}$, $-\frac{dH}{dt}$, which we denote as heretofore by ψ.

Any system of closed currents in a uniform conducting shell derived from any arbitrarily assigned current function can be caused by the electromotive forces due to some varying magnetic system, including the currents themselves, with the associated function belonging to those forces.

If, however, it be prescribed that F, G, H relate exclusively to the system itself, that is, if it be a system decaying in its own field in the absence of external forces, the equations (A) cannot be satisfied unless σ, or the thickness of the shell, be suitably chosen at every point. For in this case $\frac{dF}{dt}$, $\frac{dG}{dt}$, and $\frac{dH}{dt}$, and therefore also $\frac{d\psi}{dx}$, &c., are expressible as linear functions of $\frac{d\phi}{dt}$.

Assuming them so expressed, and ϕ and σ and h arbitrarily given, the equations (A) express two independent conditions which the single unknown quantity $\frac{d\phi}{dt}$ has to satisfy at each point on the surface. This is not generally possible. But if h,

the thickness of the shell, or σ, be also disposable, then σ or h and $\frac{d\phi}{dt}$ are determined by those equations.

We may conceive a system in which these conditions are satisfied, and continue to be so during the whole process of decay, in which therefore equations (A) hold true when differentiated according to t. We might call such a mode of decay of a system of currents a *natural decay*. The complete solution of any such problem involves the determination of $\frac{d\phi}{dt}$ or ϕ as a function of the time, which can only be effected in special cases.

426.] We will here consider the case in which the currents are such and the resistance is so adjusted at each point of the conducting shell as that all the currents decay *pari passu*, bearing at every instant during the process the same proportion to one another.

If this be the case, we shall have

$$\frac{du_s}{dt} = -\lambda u_s, \quad \frac{dv_s}{dt} = -\lambda v_s, \quad \frac{dw_s}{dt} = -\lambda w_s,$$

where λ is a constant proportional to the resistance. And the same law must hold for all linear functions of u_s, v_s, w_s, so that

$$\frac{dF}{dt} = -\lambda F, \text{ \&c.,} \quad \frac{d\phi}{dt} = -\lambda \phi, \quad \text{and} \quad \frac{d\Omega}{dt} = -\lambda \Omega;$$

and since T is a quadratic function of u_s, v_s, w_s,

$$\frac{dT}{dt} = -2\lambda T,$$

expressing the rate at which heat is generated in the decaying system. Also if F_1, G_1, &c. denote the initial values of those functions when $t = 0$, we shall have at time t

$$F = F_1 \epsilon^{-\lambda t}, \text{ \&c., and } T = T_1 \epsilon^{-2\lambda t}.$$

The constant λ is called the *modulus* of the system.

Any system of currents in a shell which has this property of decaying proportionally in its own field shall be defined to be a *self-inductive system*.

427.] We proceed to investigate the conditions that a system may be *self-inductive*, and its properties when it is so.

Let us denote by ψ the associated function, as defined in Art. 420, to the vector whose components are $-\frac{dF}{dt}$, $-\frac{dG}{dt}$, and $-\frac{dH}{dt}$. Also let χ be the associated function to $-F$, $-G$, and $-H$.

If the system be self-inductive,

$$\frac{dF}{dt} = -\lambda F, \text{ \&c., and } \frac{d\psi}{dx} = -\lambda \frac{d\chi}{dx}, \text{ \&c.}$$

The equations (A) become in this case

$$\left.\begin{aligned} \frac{\sigma}{h} u_s &= \lambda\left(F + \frac{d\chi}{dx}\right), \\ \frac{\sigma}{h} v_s &= \lambda\left(G + \frac{d\chi}{dy}\right), \\ \frac{\sigma}{h} w_s &= \lambda\left(H + \frac{d\chi}{dz}\right), \end{aligned}\right\}$$

and therefore

$$\frac{F + \frac{d\chi}{dx}}{u_s} = \frac{G + \frac{d\chi}{dy}}{v_s} = \frac{H + \frac{d\chi}{dz}}{w_s} = \frac{\sigma}{h\lambda}. \quad . \quad . \quad . \quad . \quad \text{(B)}$$

428.] Now if for any conducting shell we choose an arbitrary current function ϕ, the quantities u_s, v_s, w_s, F, G, H, and χ are all determinate at every point as functions of ϕ. The vector whose components are $F + \frac{d\chi}{dx}$, $G + \frac{d\chi}{dy}$, $H + \frac{d\chi}{dz}$ is necessarily in the tangent plane at every point, because

$$lF + mG + nH = -\frac{d\chi}{d\nu}$$

by definition, but it is not generally in the same line with the resultant current. But unless it be in the same line with the resultant current the equations (B) cannot co-exist, and therefore the system cannot be self-inductive.

The equations (B) then express the condition which the current function ϕ must satisfy, in order that the system of currents derived from it in the given shell may be capable of being made self-inductive. They express only one condition, namely that two lines, both ascertained to be in a given plane, shall coincide. It may be expressed by the single partial differential equation

$$\left(F + \frac{d\chi}{dx}\right)\frac{d\phi}{dx} + \left(G + \frac{d\chi}{dy}\right)\frac{d\phi}{dy} + \left(H + \frac{d\chi}{dz}\right)\frac{d\phi}{dz} = 0$$

at each point of the surface. This is a partial differential equation in ϕ only, because F, G, H and χ are determined if ϕ be given.

429.] As there are as many disposable quantities, namely the values of ϕ at all points on the sheet, as there are conditions to be fulfilled, namely, the above partial differential equation at every point, we may assume that for every surface S there is at least one function ϕ which satisfies the condition. We shall see later that if S be a sphere, and in certain other special cases, there are many. If ϕ be any function which fulfils this condition, then F, G, H, and χ, derived, from it satisfy equations (B),

$$\frac{F+\frac{d\chi}{dx}}{u_s}=\frac{G+\frac{d\chi}{dy}}{v_s}=\frac{H+\frac{d\chi}{dy}}{w_s}=\frac{\sigma}{h\lambda}=Q, \text{ suppose,}$$

at each point on S. But Q generally varies from point to point on the surface.

In order to make the system with ϕ so chosen actually *self-inductive*, we must so choose h as to satisfy equations (B) or

$$\frac{\sigma}{h}=\lambda Q \text{ at every point.}$$

If σ be constant, this determines the relative thickness at every point which the shell S must have in order that it may be self-inductive with the current function ϕ.

If for any system of currents on a surface the tangential component of vector potential coincides with the current at every point, the system can be made self-inductive by properly assigning the thickness of the shell. For $\frac{d\chi}{d\nu}$ is made equal and opposite to the normal component of vector potential by definition, and therefore the vector whose components are $F+\frac{d\chi}{dx}$, &c., is the tangential component of vector potential.

Examples of Self-inductive Systems.

430.] (1) a sphere of radius a.

Let ϕ be a spherical harmonic of any one order, as n. Then

$$u = \frac{z}{a}\frac{d\phi}{dy} - \frac{y}{a}\frac{d\phi}{dz}, \quad v = \&c.,$$

and therefore u, v, w are spherical harmonics of order n. And therefore, by Art. 66,

$$F = \frac{4\pi a}{2n+1} u,$$

$$G = \frac{4\pi a}{2n+1} v,$$

$$H = \frac{4\pi a}{2n+1} w,$$

and $F : G : H :: u : v : w$ at all points on the surface.

The vector potential then coincides with the current, and $\chi = 0$, $\psi = 0$, and the shell, if of uniform material and uniform thickness, is self-inductive with $\phi = AY_n$ and A constant.

(2) S a solid of revolution about the axis of z, and ϕ any function of z only which makes $\frac{d\phi}{dz}$ always of the same sign throughout S. For the currents are in circles in planes parallel to that of x, y, and so evidently are the lines of resultant vector potential, and therefore the vector potential coincides with the current at every point, and $\frac{d\phi}{dz}$ being of the same sign throughout S the currents are in the same direction round all the circles, and so therefore are the lines of vector potential. In this case $\chi = 0$ and $\psi = 0$.

(3) In any case if ϕ be a function of z only, and if χ, being the associated function derived from it, $\frac{d\chi}{dz} = 0$, the system is self-inductive.

For both the resultant current and the resultant of $F + \frac{d\chi}{dx}$, &c., are in this case in the intersection of the tangent plane with a plane parallel to that of xy.

Hence
$$\frac{F + \frac{d\chi}{dx}}{u_s} = \frac{G + \frac{d\chi}{dy}}{v_s},$$

and
$$H + \frac{d\chi}{dz} = 0, \quad w_s = 0.$$

An example of this is given later, namely, an ellipsoid with

the axis of z for one of its axes of figure. And it is shown that in this case the thickness of the shell at any point must be proportional to the perpendicular from the centre on the tangent plane at the point.

On Self-inductive Systems generally.

431.] We have seen that in self-inductive systems every linear function of u, v, and w decays according to the same law. Now Ω, the magnetic potential, is such a linear function. Therefore the variation of Ω due to resistance alone, there being no variation of the external field, is given by $\frac{d\Omega}{dt} = -\lambda\Omega$.

But the variation of Ω, due to variation of the external field in the absence of resistance, is given by

$$\frac{d\Omega}{dt} = -\frac{d\Omega_0}{dt},$$

Ω_0 being the magnetic potential of the external field.

Therefore for the whole time variation of Ω, we have

$$\frac{d\Omega}{dt} = -\lambda\Omega - \frac{d\Omega_0}{dt},$$

from which Ω may be determined as a function of t whenever the law of variation of the external field is given.

EXAMPLE I. Let $\frac{d\Omega_0}{dt}$ be constant. Then the equation becomes

$$\frac{d\Omega_0}{dt} + \lambda\Omega = C,$$

or

$$\Omega = \frac{C}{\lambda}(1 - \epsilon^{-\lambda t}).$$

If we make λt infinitely small while Ct remains finite, this represents the ideal case of so-called impulsive currents, that is, a system of finite currents supposed to be created in an infinitely short time, and Ct represents the impulse. In this case the equation gives $\Omega = Ct$, and Ω is independent of the resistance.

If, on the other hand, we make λt very great compared with unity, as we always may do by sufficiently increasing the resistance, or the time, we obtain $\Omega = \frac{C}{\lambda}$. That is, Ω varies inversely

as the resistance. The resistance in this case plays a part analogous to that of mass in the motion of a material system under finite forces from rest.

Suppose, for instance, there be several conducting shells, and a magnetic system external to all of them, whose magnetic potential is made to vary so that $\frac{d\Omega_0}{dt}$ is constant. And suppose that the systems of currents generated in the shells are self-inductive; then, according to the result last obtained, the currents in the shells will, as the time increases, become inversely proportional, *cæteris paribus*, to the resistances. This result agrees with the assumption with which we started, that induced currents may be regarded as existing in conductors only, because, although no substance is a perfect conductor or a perfect insulator, the resistance in so-called insulators bears a very high ratio to that in metals.

EXAMPLE II. Let the potential of the external magnetic field on or within the shell S be given by

$$\Omega_0 = A\cos\kappa t,$$

where κ is constant, and A constant as regards time, but having different values at different points. Then at any internal point we have, if the system of currents be self-inductive,

$$\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt} + \lambda\Omega = 0.$$

Let $\lambda = \kappa\cot a.$

Then $\frac{d\Omega_0}{dt} + \frac{d\Omega}{dt} + \kappa\cot a\,\Omega = 0,$

and $\Omega_0 = A\cos\kappa t.$

To solve this assume

$$\Omega = (\cos\kappa t + q\sin\kappa t)A'.$$

Then we have, neglecting constant factors,

$$-A\sin\kappa t - A'\sin\kappa t + qA'\cos\kappa t + A'\cot a(\cos\kappa t + q\sin\kappa t) = 0.$$

And therefore $q = -\cot a,$

and $A + A' + A'\cot^2 a = 0.$

or $A' = -A\frac{1}{1+\cot^2 a} = -A\sin^2 a.$

Therefore at any internal point

$$\begin{aligned}\Omega &= -A\sin^2 a(\cos\kappa t - \cot a \sin\kappa t)\\ &= A\sin a \sin(\kappa t - a).\end{aligned}$$

And for the whole magnetic potential at any point within S

$$\begin{aligned}\Omega_0 + \Omega &= A\{\cos\kappa t + \sin a \sin(\kappa t - a)\}\\ &= A\cos a \cos\overline{\kappa t - a}.\end{aligned}$$

The field is diminished in intensity in the proportion $\cos a : 1$, and retarded in phase by $\frac{a}{2\pi}$ of a complete period*.

432.] If λ be very great compared with κ, a becomes nearly zero, and $\sin a = a = \frac{\kappa}{\lambda}$. In this case the internal field has the same intensity, because $\cos a = 1$, and nearly the same phase, as the external field. This is the state of things to which we approximate as we diminish indefinitely the thickness, or increase indefinitely the specific resistance, of the conducting shell. Now we may conceive a solid conductor to consist of a number of infinitely thin shells successively enclosing one another, and apply the formulae above obtained to each shell. Let the shells be of such thickness that each is self-inductive with the given currents. Then the same phase is reached in an inner shell at a time $\frac{a}{\kappa}$, that is $\frac{1}{\lambda}$, later than in the shell immediately outside of it. The ratio which h, the thickness of the outer shell, bears to this difference of time, is in the limit *the velocity with which a disturbance of the type in question penetrates the solid.* This velocity is then λh, that is $\frac{\sigma}{Q}$ in the notation of Art. 429. This relation holds true so long only as we can neglect the inductive action of inner shells upon outer ones.

It is assumed in the above investigation that the system of currents induced in the shell at any instant by the variation of the field is self-inductive. This, if true at any instant, is true

* This problem is treated by Professor Larmor, Phil. Mag., 1884, for the special case of a spherical sheet.

at every instant, because the values of Ω at all points in or within the shell are multiplied by the same factor $\cos \kappa t$.

433.] Any two or more self-inductive systems of currents may co-exist on a conducting shell; and if they have the same modulus, or value of λ, they combine to form one self-inductive system with that same value of λ. For let F_1 &c. relate to one system, and F_2 &c. to another. Then at any point on the surface

$$\frac{\sigma}{h} = \lambda \frac{F_1 + \frac{d\chi_1}{dx}}{u_1} = \lambda \frac{F_2 + \frac{d\chi_2}{dx}}{u_2}$$

$$= \lambda \frac{F_1 + F_2 + \frac{d(\chi_1 + \chi_2)}{dx}}{u_1 + u_2},$$

and therefore $u_1 + u_2$ is the component of superficial current in a self-inductive system with λ for modulus. And so on for any number of systems.

434.] We can now treat the following case. Let a conducting shell whose surface is a solid of revolution, revolve with uniform angular velocity ω about its axis of figure in a field of uniform magnetic force, $-P$, at right angles to that axis. Let the axis of rotation be that of y, and the direction of the force that of x. Let us take any plane through the axis fixed in the conductor for the plane of reference, and let time be measured from an epoch at which the plane of reference coincides with that of xy. Then if Q be any point on or within the conductor distant r from the axis, and such that a plane through the axis and through Q makes the angle θ with the plane of reference, the potential at Q of the external field is $Pr\cos(\omega t + \theta)$.

By the change of position of the field relative to the shell we have induced on the shell at any instant a system of currents which is symmetrical with respect to that particular plane through the axis which coincides at that instant with the plane of xy. Let us suppose that the system of currents induced at any instant by the rotation is a self-inductive system. We have then a series of self-inductive systems successively created, symmetrical with regard to successive planes through the axis fixed in the conductor.

We have then, to find the potential at Q of the induced currents,
$$\frac{d\Omega_0}{dt}+\frac{d\Omega}{dt}+\lambda\Omega=0,$$
and $\Omega_0 = Pr\cos(\omega t+\theta)$
$$\frac{d\Omega_0}{dt}=-P\omega r\sin(\omega t+\theta);$$
whence we obtain
$$\Omega = Pr\sin a\sin(\omega t+\theta-a),$$
$$\Omega_0+\Omega = Pr\cos a\cos(\omega t+\theta-a),$$
a having the value $\cot^{-1}\frac{\lambda}{\omega}$.

435.] We may calculate in this case the mechanical work done per unit of time in turning the conducting shell. For this is the same as would be done if the currents were at each instant replaced by the corresponding system of magnetic shells over the surface with strength ϕ. It is therefore
$$\iint\phi\frac{d}{dt}\cdot\frac{d\Omega_0}{dv}dS,$$
if Ω_0 be the magnetic potential of the given field, and therefore $\frac{d\Omega_0}{dv}dS$ the flux through the elementary area dS of the magnetic induction of the field. Now
$$\frac{d\Omega_0}{dv}=P\sin\beta\cos(\omega t+\theta),$$
where β is the angle which the normal to dS makes with the axis of figure, and
$$\frac{d}{dt}\frac{d\Omega_0}{dv}=-P\omega\sin\beta\sin(\omega t+\theta).$$
Also
$$\phi = PA\cos(\omega t+\theta-a),$$
where A is a constant depending on the form of the surface. Hence
$$\iint\phi\frac{d}{dt}\frac{d\Omega}{dv}dS$$
is proportional to
$$\int_0^{2\pi}d\theta\sin(\omega t+\theta)\cos(\omega t+\theta-a),$$
that is to $\sin a$. It is proportional, as in the case of the single closed circuit treated of in Art. 355, to the sine of the retardation of phase.

436.] We might in any of the preceding cases suppose a core of soft iron within the shell and separated from it by a thin lamina of non-conducting matter. The magnetic potential of the field in which the core is placed is, as we have seen, of the form $A \cos a \cos(\omega t - a)$. The core will be magnetised by induction but with a retardation of phase as compared with the field in which it is placed*.

If we were to assume that the magnetisation bears a constant ratio to the magnetic induction, we should have for the determination of Ω an equation of the form

$$\frac{d\Omega}{dt} + \lambda\Omega + C\cos(\kappa t + x) = 0$$

where C is constant, which would lead to a solution of the same form as before.

As the relation between the induced magnetisation and the magnetic induction in soft iron is not perhaps sufficiently established, it may not be safe to draw any but the following general conclusions, viz. (1) the magnetisation of the iron will be always retarded in phase as compared with the field in which it is placed, and therefore as compared with the external field; (2) if this retardation be not very great the magnetic field due to the core will at all points on the shell be of the same sign as the external field, and the effect of the core will be to *increase* the currents induced in the shell.

Further, as soft iron, although magnetisable, is a conductor, there would, were the surface of the soft iron continuous, be also induced currents on it which would create a magnetic field of the opposite sign to that of the induced magnetisation, and so tend to *diminish* the induced currents in the shell. But this may be obviated by making the core consist of insulated iron wires running in directions at right angles to the currents in the shell. This is usually done in forming the core of the armature of a dynamo-machine.

437.] The treatment of special cases is reserved generally for the following chapter, but in order to elucidate a general prin-

* We assume here and throughout the chapter that the oscillations of the field are not too rapid.

ciple of some importance, we will again anticipate one of the results there proved. It is shown, namely in that chapter, that if currents of the type $\phi = Az$, where A is constant, be generated in an ellipsoidal shell of which one axis coincides with z, the system will be self-inductive when the thickness of the shell at any point is proportional to the perpendicular, ϖ, from the centre on the tangent plane at the point. The component currents per unit of area are then

$$u = \frac{Ay}{b^2}, \quad v = -\frac{Ax}{a^2}, \quad w = 0;$$

or if

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2};$$

$$2u = \frac{dS}{dy}\frac{d\phi}{dz} - \frac{dS}{dz}\frac{d\phi}{dy} = \frac{2Ay}{b^2}.$$

If the equation to the given ellipsoid be

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then

$$\frac{x^2}{r^2a^2} + \frac{y^2}{r^2b^2} + \frac{z^2}{r^2c^2} = 1, \text{ or } S = r^2,$$

is the equation to a similar, similarly situated, and concentric ellipsoid S', whose linear dimensions are to those of the given ellipsoid as $r : 1$. And we shall suppose $r < 1$, and S' an inner ellipsoid.

It can now be shown that if we form on the ellipsoid S' a shell of uniform material similar to the given shell S, then the generation of the given system of currents on the given shell S will cause by induction a system of currents of the corresponding type, but in the reverse direction, on the inner shell S'. For let F, G, H be the components of vector potential of the given currents in the outer shell, χ their associated function. And let F', G', H' be the corresponding functions for the induced currents on the inner shell.

Then, since the given system of currents is self-inductive, we have

$$\lambda\left(F + \frac{d\chi}{dx}\right) = \frac{\sigma}{h}u_s = \sigma u,$$

at all points on S, λ being the modulus. Now by the definition of χ,

$$\nabla^2\left(F + \frac{d\chi}{dx}\right) = 0,$$

at all points within S. Also, u being the function $\frac{Ay}{b^2}$, $\nabla^2 u = 0$ at all points within S.

Therefore
$$\lambda\left(F+\frac{d\chi}{dx}\right)=\sigma u$$
at all points within the ellipsoid S, and therefore at all points on the ellipsoid S'.

That is,
$$-\frac{dF}{dt}-\frac{d\psi}{dx}=\sigma u,$$
at all points on S', substituting $-\frac{dF}{dt}$ for λF and $-\psi$ for $\lambda\chi$, as in Art. 427.

That is, the continuous increase of the given currents on S acts as an electromotive force tending to produce the reverse currents $-u$, $-v$, and $-w$ on S'.

But since this system of currents on S' is self-inductive, its own self-induction will not cause currents of any other type to appear in S'. This type of currents will therefore be induced with the opposite sign to those in the outer shell.

438.] We have dealt with the case of an ellipsoid only. But the same method may be extended thus. Let S be any homogeneous function of positive degree in x, y, and z, and $S = 1$ a given surface.

Then we may divide the space within S into a series of similar concentric and similarly situated shells, each being between two surfaces such as $S = c$ and $S = c+dc$, where $c < 1$.

Let us suppose that in each of these shells, if a conducting shell, a system of currents with ϕ for current function would be self-inductive. Let an inner conducting shell be so formed. Let an outer shell be formed on S, i.e. between $S = c$ and $S = c-dc$, and let the given type of currents be generated in it. Then it can be proved by the same method as we employed in the case of the ellipsoid that a system of currents of the type ϕ would be generated by induction in the inner shell, provided only that the functions u and v, or
$$\frac{dS}{dz}\frac{d\phi}{dy}-\frac{dS}{dy}\frac{d\phi}{dz}, \text{ \&c.,}$$

be of positive degree, and $\nabla^2 u = 0$, and similarly $\nabla^2 v = 0$ at all points within S.

The Effect of Resistance; Solid Conductors.

439.] As the superficial currents decay by resistance, they no longer act as a complete magnetic screen to the internal portions of the solid. These accordingly become subject to the influence of a varying external magnetic field, and currents are excited in them also, so that in time the whole solid becomes pervaded by currents; and this time is perhaps generally so short that the process is sensibly instantaneous. The laws of this process are expressed by the equation,

$$\left.\begin{aligned} \frac{d(F_0+F)}{dt} + \frac{d\psi}{dx} - \sigma u &= 0 \\ \frac{d(G_0+G)}{dt} + \frac{d\psi}{dy} - \sigma v &= 0 \\ \frac{d(H_0+H)}{dt} + \frac{d\psi}{dz} - \sigma w &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

with the bounding condition

$$lu + mv + nw = 0.$$

In dealing with problems of this class it is frequently more convenient to retain the variables F, G, and H. For instance, if S be a solid sphere of uniform material, and the external magnetic system be due to closed currents on spherical surfaces concentric with S, we shall have no statical potential ψ, and the equations become

$$\begin{aligned} \frac{dF_0}{dt} + \frac{dF}{dt} &= -Ru \\ &= -\frac{R}{4\pi}\nabla^2 F. \end{aligned}$$

Assuming, as in the second example of Art. 431,

$$F_0 = A \cos\kappa t\, Y_n,$$

the problem admits of solution *. The treatment of this class of cases is reserved for the next chapter.

* See the memoir by Professor Larmor, above quoted. Also a memoir by Professor Niven, R. S. Phil. Trans., 1881.

Also, as above shown, for any current sheet $S = c$,

$$F = \iint \frac{1}{r}\left(m\frac{d\phi}{dy} - m\frac{d\phi}{dz}\right) dS$$

$$= \iint \phi\left(m\frac{d}{dz} - n\frac{d}{dy}\right)\frac{1}{r}\, dS,$$

with corresponding values for G and H, and also

$$\Omega = \iint \phi \frac{d}{d\nu}\frac{1}{r}\, dS.$$

445.] We have already, in Chap. XVIII, investigated the magnetic field in the neighbourhood of an infinite rectilinear current, and we now proceed to do the same for the field due to certain given systems of currents on certain closed surfaces.

This is completely determined when F, G, H are known at every point, also since at all points not situated on the surface the magnetic force is derivable from a potential Ω the investigation will include the determination of Ω at all such points.

For example, suppose the sheet to be spherical with radius a, then the values of u, v, w are given by the equations

$$u = \frac{z}{a}\cdot\frac{d\phi}{dy} - \frac{y}{a}\cdot\frac{d\phi}{dz}, \quad v = \frac{x}{a}\cdot\frac{d\phi}{dz} - \frac{z}{a}\cdot\frac{d\phi}{dx}, \quad w = \frac{y}{a}\cdot\frac{d\phi}{dx} - \frac{x}{a}\cdot\frac{d\phi}{dy},$$

where ϕ is the current function.

F, G, H must satisfy the potential conditions

$$\nabla^2 F = \nabla^2 G = \nabla^2 H = 0$$

everywhere except at the surface, must be everywhere finite and continuous, and satisfy the condition of no convergency

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0$$

at all points.

At the surface, if F', G', H' denote the values of F, G, H within the surface and F, G, H without,

$$\frac{dF}{dr} - \frac{dF'}{dr} = -4\pi u, \quad \frac{dG}{dr} - \frac{dG'}{dr} = -4\pi v, \quad \frac{dH}{dr} - \frac{dH'}{dr} = -4\pi w.$$

Since F, G, H are potential functions it follows that the solution of the problem is unique, and therefore that any solution is the general solution.

Now all the equations in F, G, H may be satisfied if

$$F=\left(\frac{z}{a}\cdot\frac{d}{dy}-\frac{y}{a}\cdot\frac{d}{dz}\right)P,\quad G=\left(\frac{x}{a}\cdot\frac{d}{dz}-\frac{z}{a}\cdot\frac{d}{dx}\right)P,$$

$$H=\left(\frac{y}{a}\cdot\frac{d}{dx}-\frac{x}{a}\cdot\frac{d}{dy}\right)P,$$

provided P be everywhere finite and continuous, and $\nabla^2 P$ be zero at all points not on the surface, and on the surface

$$\frac{dP}{dr}-\frac{dP'}{dr}=-4\pi\phi,$$

where P and P' are external and internal values of P.

Also at all points not on the surface

$$\frac{d\Omega}{dx}=\frac{dG}{dz}-\frac{dH}{dy}=\frac{x}{a}\nabla^2 P-\frac{1}{a}\cdot\left(x\frac{d^2P}{dx^2}+y\frac{d^2P}{dx\,dy}+z\frac{d^2P}{dx\,dz}\right)-\frac{2}{a}\cdot\frac{dP}{dx}$$

$$=-\frac{1}{a}\cdot\left\{\frac{d}{dx}\left(x\frac{dP}{dx}+y\frac{dP}{dy}+z\frac{dP}{dz}\right)+\frac{dP}{dx}\right\};$$

$$\because\quad \nabla^2 P=0;$$

$$\therefore\quad \Omega=-\frac{1}{a}\left\{r\frac{dP}{dr}+P\right\}=-\frac{1}{a}\cdot\frac{d}{dr}(Pr).$$

It follows, therefore, that the field is completely determined by mere differentiation when the quantity P is known, and this quantity from the conditions which it satisfies is the potential of matter of density ϕ over the spheres surface.

We might have treated the problem otherwise, for by what has been already proved we know that

$$F=\iint\phi\left(\frac{y}{a}\cdot\frac{d}{dz}-\frac{z}{a}\cdot\frac{d}{dy}\right)\frac{dS}{r},$$

and

$$\Omega=\iint\phi\frac{d}{dr}\cdot\frac{dS}{r},$$

leading to the same results as above when

$$P=\iint\frac{\phi}{r}\cdot dS.$$

446.] The conditions to be satisfied by P indicate that the most general form which can be assumed by it is a series of spherical harmonic functions of the type

$$M\left(\frac{a}{r}\right)^{i+1}Y_i$$

without the spherical sheet, and

$$M\left(\frac{r}{a}\right)^i Y_i$$

within the sheet, where M is some constant, and Y_i is a surface harmonic of the degree i.

The surface condition

$$\frac{dP}{dr} - \frac{dP'}{dr} = -4\pi\phi$$

gives, corresponding to each term of the degree i,

$$\phi = \frac{2i+1}{4\pi} \cdot \frac{M}{a} Y_i$$

for the current function on the sheet.

And the equation

$$\Omega = -\frac{1}{a}\frac{d}{dr}(Pr)$$

gives, if Ω and Ω' be the value of Ω at points without and within the sheet respectively,

$$\Omega = i\frac{M}{a}\left(\frac{a}{r}\right)^{i+1} Y_i, \quad \text{and} \quad \Omega' = -(i+1)\frac{M}{a}\left(\frac{r}{a}\right)^i Y_i.$$

If, as is sometimes more convenient, the system be determined from the form of ϕ, and we assume

$$\phi = AY_i$$

on the sheet, we have

$$P = \frac{4\pi a}{2i+1} A\left(\frac{a}{r}\right)^{i+1} Y_i, \quad P' = \frac{4\pi a}{2i+1} A\left(\frac{r}{a}\right)^i Y_i;$$

$$\Omega = \frac{4\pi i}{2i+1} A\left(\frac{a}{r}\right)^{i+1} Y_i, \quad \Omega' = -\frac{i+1}{2i+1} 4\pi A\left(\frac{r}{a}\right)^i Y_i.$$

If Y_i be the zonal-surface harmonic with axis z of the first order, and therefore ϕ be of the form $A\cos\theta$ where θ is the angular distance from the axis z, then outside of the sheet

$$\Omega = \frac{4\pi}{3}\frac{Aa^2}{r^2} Y_i = \frac{4\pi}{3} \cdot \frac{Aa^2}{r^2}\cos\theta,$$

and inside of the sheet

$$\Omega' = -\frac{8\pi}{3}\frac{A}{a} r\cos\theta.$$

Or the external field is that of a small magnet at the centre of the sheet with axis along that of z and moment $\frac{4\pi}{3}\cdot Aa^2$.

And the internal field is one of constant force parallel to z and equal to $\frac{8\pi A}{3a}$.

Since ϕ is a function of θ it follows that the resultant current referred to unit length is at every point perpendicular to the meridian and equal to

$$-\frac{1}{a}\cdot\frac{d\phi}{d\theta} \quad \text{or} \quad \frac{A}{a}\sin\theta,$$

whence the total quantity crossing any meridian

$$= A\int_0^\pi \sin\theta\, d\theta = 2A.$$

If the current be in a wire of uniform transverse section coiled round the sphere's surface, and n be the total number of coils, the number of windings from the pole to latitude θ is

$$\frac{n}{2}(1-\sin\theta).$$

447.] Again, let $\phi = AY_2$ where Y_2 is the superficial zonal-spherical harmonic of the second degree with axis z, and therefore equal to $\frac{3\cos^2\theta}{2}-\frac{1}{2}$, where θ is the angular distance from z.

In this case we have outside the sheet

$$\Omega = \frac{8\pi}{5}A\frac{a^3}{r^3}\cdot\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right) = \frac{4\pi Aa^3}{5r^5}(2z^2-x^2-y^2),$$

and within the sheet

$$\Omega' = -\frac{12\pi}{5}\frac{Ar^2}{a^2}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right) = -\frac{6\pi A}{5a^2}(2z^2-x^2-y^2);$$

or within the sheet the potential is that of a homogeneous function of the second degree.

Since ϕ is a function of θ only it follows, as in the last case, that the current referred to unit length is in parallels of latitude and equal to

$$-\frac{1}{a}\cdot\frac{d\phi}{d\theta} \quad \text{or} \quad \frac{3A}{2a}\sin 2\theta.$$

If produced by coiling a wire round the sphere, we must remember that the direction of the coils must be reversed on crossing the equator, because of the change of sign in z, that is in $\sin 2\theta$.

The density of the coils in different latitudes is easily calculated as in the last case (see Maxwell's *Electricity*, vol. II. Chap. VII. —

448.] If the sheet were an infinite plane, we might treat it as a particular case of the spherical sheet by supposing the radius to be infinitely increased, but it is more interesting to investigate it independently.

Taking the plane of the sheet for that of x, y, the several equations become

$$u = \frac{d\phi}{dy}, \quad v = -\frac{d\phi}{dx}, \quad w = 0,$$

and therefore $H = 0$,

$$\nabla^2 F = \nabla^2 G = 0.$$

F and G everywhere continuous and finite, as also their differential coefficients, except at the sheet or when $z = 0$, in which case

$$\frac{dF}{dz} - \frac{dF'}{dz} = -4\pi u, \quad \frac{dG}{dz} - \frac{dG'}{dz} = -4\pi v;$$

and since by symmetry $\frac{dF}{dz} = -\frac{dF'}{dz}$, and $\frac{dG}{dz} = -\frac{dG'}{dz}$, these last equations become

$$\frac{dF}{dz} = -2\pi u, \quad \frac{dG}{dz} = -2\pi v.$$

The condition of no convergence gives us

$$\frac{dF}{dx} + \frac{dG}{dy} = 0.$$

All these equations are satisfied by the assumption

$$F = \frac{dP}{dy}, \quad G = -\frac{dP}{dx},$$

provided $\nabla^2 P = 0$ everywhere except upon the sheet, P be everywhere finite and continuous as well as its differential coefficients, except upon the sheet, and at the sheet

$$\frac{dP}{dz} = -2\pi\phi.$$

As in the case of the sphere, therefore we infer that P is the potential of matter of surface density ϕ upon the sheet.

If Ω be the magnetic potential at any point not on the sheet then, since

$$\frac{d\Omega}{dx} = \frac{dG}{dz} - \frac{dH}{dy},$$

we get

$$\frac{d\Omega}{dx} = -\frac{d^2 P}{dx\,dz},$$

and

$$\Omega = -\frac{dP}{dz}.$$

The most general expression for P in this case is a series of terms of the form

$$P = e^{-mz}\psi(x, y)$$

on the positive side of the sheet, and

$$P = e^{-mz'}\psi(x, y)$$

on the negative side of the sheet, where z' is measured in the opposite direction to z, i.e. away from the sheet on the negative side, and ψ satisfies the equation

$$(\nabla_1^2 + m^2)\psi = 0,$$

where ∇_1^2 stands for $\frac{d^2}{dx^2} + \frac{d^2}{dy^2}$.

Or if ϕ be expressed in a series of terms of the form $\psi(x, y)$ where

$$(\nabla_1^2 + m^2)\psi = 0,$$

P will be a series of terms of the form

$$\frac{2\pi}{m}e^{-mz}\psi(x, y) \quad \text{and} \quad \frac{2\pi}{m}e^{-mz'}\psi(x, y),$$

on the positive and negative sides respectively*.

* The possibility of expressing ϕ as required in the text follows from the possibility of expressing any function of the position of a point on a spherical surface in a series of surface spherical harmonics.

For if u be one of the harmonic terms of such a function of order (i), we know that

$$\frac{d^2 u}{d\theta^2} + \cot\theta\frac{du}{d\theta} + \frac{1}{\sin^2\theta}\cdot\frac{d^2 u}{d\phi^2} + i(i+1)u = 0.$$

If (a) be the radius of the sphere and be very large, then in order that x and y may be finite θ must be very small, and therefore, unless i be very large, this equation becomes $\frac{d^2 u}{d\phi^2} = 0$, but if i be infinitely great we have

$$\frac{d^2 u}{d\phi^2} + i^2 u\sin^2\theta = 0;$$

449.] Finally, consider the case of a shell in the form of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Here if ϖ be the perpendicular from the centre on the tangent plane at x, y, z,

$$l=\frac{\varpi x}{a^2}, \quad m=\frac{\varpi y}{b^2}, \quad n=\frac{\varpi z}{c^2}.$$

Let ϕ be equal to $-Az$.

Then $$u=A\varpi\frac{y}{b^2}, \quad v=-A\varpi\frac{x}{a^2}, \quad w=0$$

(u, v, w being currents referred to unit of length), and therefore $H=0$.

The quantities F and G must satisfy the equations $\nabla^2F=0$, $\nabla^2G=0$ everywhere except on the sheet, must be everywhere finite and continuous, as well as their differential coefficients, except on the sheet, where these last satisfy the equations

$$\frac{dF}{d\nu}-\frac{dF'}{d\nu}=-4\pi A\varpi\frac{y}{b^2}, \quad \frac{dG}{d\nu}-\frac{dG'}{d\nu}=4\pi A\varpi\frac{x}{a^2}.$$

Now we may prove, as in Chap. XVII, Art. 318, above, that the conditions determining F, F', G, G' may be satisfied by taking

$$A_1\frac{y}{b^2} \quad \text{and} \quad A_1\frac{y}{b^2}\frac{\frac{d\Phi}{db^2}}{\frac{d\Phi_0}{db^2}} \quad \text{for } F' \text{ and } F,$$

and $$-A_2\frac{x}{a^2} \quad \text{and} \quad -A_2\frac{x}{a^2}\frac{\frac{d\Phi}{da^2}}{\frac{d\Phi_0}{da^2}} \quad \text{for } G' \text{ and } G,$$

where

$$\Phi=\int_\lambda^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \quad \Phi_0=\int_0^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

whence, if $a\sin\theta=r$, and $x=r\cos\phi$, $y=r\sin\phi$;

$$\frac{d^2u}{d\phi^2}=r^2\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right),$$

and therefore $$\nabla_1^2u+\frac{i^2}{a^2}u=0;$$

therefore $$(\nabla_1^2+m^2)u=0,$$

where m is the reciprocal of the infinitely large radius of the sphere divided by the infinitely large order of the spherical harmonic.

being the positive root of the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1.$$

For, by reasoning as in the chapter referred to, we have

$$\frac{dF}{dv}-\frac{dF'}{dv}=\frac{A_1 y}{ab^3c}\cdot\frac{\varpi}{\frac{d\Phi_0}{db^2}},\quad \frac{dG}{dv}-\frac{dG'}{dv}=-\frac{A_2 x}{a^3bc}\cdot\frac{\varpi}{\frac{d\Phi_0}{da^2}},$$

whence, comparing with the above-written surface conditions, we get

$$A_1=-4\pi ab^3cA\frac{d\Phi_0}{db^2},\quad A_2=-4\pi a^3bcA\frac{d\Phi_0}{da^2}.$$

And therefore

$$F'=-4\pi abc\frac{d\Phi_0}{db^2}Ay,\quad G'=4\pi abc\frac{d\Phi_0}{da^2}Ax.$$

Also $$\frac{d\Omega'}{dz}=\frac{dF'}{dy}-\frac{dG'}{dx}=-4\pi abcA\left\{\frac{d\Phi_0}{db^2}+\frac{d\Phi_0}{da^2}\right\},$$

$$\frac{d\Omega'}{dy}=0,\qquad \frac{d\Omega'}{dx}=0.$$

That is to say, within the ellipsoidal shell the field is one of uniform force parallel to z and equal to

$$4\pi abcA\left\{\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{db^2}\right\}.$$

If the ellipsoid become a sphere of radius (a) the values of u and v become

$$\frac{Ay}{a}\quad\text{and}\quad -\frac{Ax}{a}.$$

And the uniform force within the sheet becomes $-\dfrac{8\pi A}{3}$, agreeing with the results already obtained*.

Of course the method here employed might have been applied to the sphere substituting $\dfrac{a^3}{r^3}$ for

$$\frac{\frac{d\Phi}{da^2}}{\frac{d\Phi_0}{da^2}}\quad\text{or}\quad\frac{\frac{d\Phi}{db^2}}{\frac{d\Phi_0}{db^2}}.$$

Also we might in the ellipsoidal case have obtained the values of F and G directly from the general equations

* The quantity A in this Art. when $a=b=c$ corresponds to $-\dfrac{A}{a}$ in Art. 446.

have seen in the last chapter that unless the current function ϕ satisfies a certain condition, this law of proportional decay will not hold true and the type of currents will not be maintained throughout the decay.

In other words, the direction of the electromotive force at each point in the sheet will not be coincident with the direction of resultant current at that point, and in cases where this condition can be satisfied, it will in most cases necessitate the existence of a distribution of free electricity over the sheet.

In spherical sheets of uniform thickness the condition can be satisfied for currents of all types without the aid of such a distribution.

Thus, in the case of the spherical sheet with the notation employed above, we have

$$\sigma u = -\frac{dF}{dt} - \frac{d\psi}{dx} \quad \text{and} \quad \frac{dF}{dr} - \frac{dF'}{dr} = -4\pi u$$

at the sheet, with corresponding equations in G and H, v and w, σ being the resistance per unit area on the sheet, whence

$$\left(z\frac{d}{dy} - y\frac{d}{dz}\right)\left\{\frac{\sigma}{4\pi}\cdot\left(\frac{dP}{dr} - \frac{dP'}{dr}\right) - \frac{dP}{dt}\right\} = -\frac{d\psi}{dx},$$

$$\left(x\frac{d}{dz} - z\frac{d}{dx}\right)\left\{\frac{\sigma}{4\pi}\cdot\left(\frac{dP}{dr} - \frac{dP'}{dr}\right) - \frac{dP}{dt}\right\} = -\frac{d\psi}{dy},$$

$$\left(y\frac{d}{dx} - x\frac{d}{dy}\right)\left\{\frac{\sigma}{4\pi}\cdot\left(\frac{dP}{dr} - \frac{dP'}{dr}\right) - \frac{dP}{dt}\right\} = -\frac{d\psi}{dz};$$

therefore $$\frac{x}{a}\cdot\frac{d\psi}{dx} + \frac{y}{a}\cdot\frac{d\psi}{dy} + \frac{z}{a}\cdot\frac{d\psi}{dz} = \frac{d\psi}{dr} = 0$$

at the sheet, and since $\nabla^2\psi = 0$ everywhere except upon the sheet, it follows that $\psi = 0$ everywhere, and therefore if $\frac{\sigma}{4\pi}$ be denoted by R we have

$$R\left(\frac{dP}{dr} - \frac{dP'}{dr}\right) - \frac{dP}{dt} = 0$$

at the sheet, and therefore if the original value of ϕ were $(A_i)_0 Y_i$ we should have

$$RA_i Y_i + \frac{a}{2i+1}\frac{dA_i}{dt}\cdot Y_i = 0.$$

Therefore $$A_i = (A_i)_0 e^{-\frac{(2i+1)}{a}Rt},$$

where A_i is the value of A_i at any time t from the first establishment of the currents.

Whence it follows that F, G, H, Ω all vary according to the law $e^{-\lambda t}$ where $\lambda = \frac{2i+1}{a}R$.

452.] For the infinite plane sheet, with the notation already employed, our equations become, at the sheet

$$\frac{d^2 P}{dy\,dt} - \frac{\sigma}{2\pi}\cdot\frac{d^2 P}{dy\,dz} = -\frac{d\psi}{dx},$$

$$-\frac{d^2 P}{dx\,dt} + \frac{\sigma}{2\pi}\cdot\frac{d^2 P}{dx\,dz} = -\frac{d\psi}{dy},$$

$$0 = -\frac{d\psi}{dz}.$$

Since $\frac{d\psi}{dz} = 0$ at the sheet, and $\nabla^2\psi = 0$ at all other points, we have $\psi = 0$: whence, writing R for $\frac{\sigma}{2\pi}$, we have

$$R\frac{dP}{dz} - \frac{dP}{dt} = 0$$

at the sheet.

If therefore the initial value of ϕ be a series of terms of the form

$$A_m \psi(x, y)$$

where

$$(\nabla_1^2 + m^2)\psi = 0,$$

the value of P at any point x, y, z at the time t will be a series of terms of the form

$$\frac{2\pi}{m} A_m e^{-m(z+Rt)}\psi(x, y),$$

and the value of ϕ at any time t will be a series of terms of the form

$$A_m e^{-mRt}\,.\,\phi_0.$$

The values of F, G, and Ω at any point are obtained from those of P by differentiation, being $\frac{dP}{dy}$, $-\frac{dP}{dx}$, and $-\frac{dP}{dz}$, as above shown.

Hence if any system of currents be established in an infinite plane sheet, and be then allowed to decay by resistance under their mutual actions, the magnetic field on the positive side of the plane at any time is the same as if the currents remained

unchanged in magnitude and direction, and the plane moved parallel to itself with the velocity R towards the negative direction.

Of course exactly similar results hold on the negative side of the sheet, the field of decay being obtained by moving the sheet in the positive direction.

453.] Finally, consider the case of the ellipsoidal sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

with the system of currents

$$u=A\frac{\varpi y}{b^2},\quad v=-A\varpi\frac{x}{a^2},\quad w=0,$$

or the current function $-Az$.

Our equations become

$$\sigma\frac{A\varpi y}{b^2}=4\pi abc\frac{d\Phi_0}{db^2}y\frac{dA}{dt}-\frac{d\psi}{dx},$$

$$\sigma\frac{A\varpi x}{a^2}=4\pi abc\frac{d\Phi_0}{da^2}x\frac{dA}{dt}+\frac{d\psi}{dy},$$

$$0=\frac{d\psi}{dz}.$$

Assume that σ varies inversely as ϖ and we get, writing $\frac{s}{\varpi}$ for σ,

$$\left(\frac{sA}{b^2}-4\pi abc\frac{d\Phi_0}{db^2}\frac{dA}{dt}\right)y=-\frac{d\psi}{dx}. \quad . \quad . \quad . \quad . \quad (1)$$

$$\left(\frac{sA}{a^2}-4\pi abc\frac{d\Phi_0}{da^2}\frac{dA}{dt}\right)x=\frac{d\psi}{dy}. \quad . \quad . \quad . \quad . \quad (2)$$

$$0=\frac{d\psi}{dz}. \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating ψ between (1) and (2), we get

$$4\pi abc\left(\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{db^2}\right)\frac{dA}{dt}=sA\left(\frac{1}{a^2}+\frac{1}{b^2}\right),$$

or $$abc\left(\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{db^2}\right)\frac{dA}{dt}=R'A\left(\frac{1}{a^2}+\frac{1}{b^2}\right),\quad R'=\frac{s}{4\pi},$$

whence we have proportional decay of the type $e^{-\lambda t}$, where

$$\lambda=-\frac{R'\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}{abc\left(\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{db^2}\right)};$$

agreeing with the result already obtained for the sphere when $a = b = c$ and $i = 1$ because $R' = aR$.

From the above equations also we get

$$\varpi\frac{x}{a^2}\frac{d\psi}{dx} + \varpi\frac{y}{b^2}\frac{d\psi}{dy} + \varpi\frac{z}{c^2}\frac{d\psi}{dz} = \frac{d\psi}{d\nu}$$
$$= 4\pi abc\frac{dA}{dt}\left\{\frac{1}{a^2}\frac{d\Phi_0}{db^2} - \frac{1}{b^2}\frac{d\Phi_0}{da^2}\right\}\varpi xy,$$

indicating an electrical distribution of determinate density on the surface.

The density of superficial distribution may be readily found from the values Mxy and

$$Mxy\,\frac{\dfrac{d\Phi}{da^2} - \dfrac{d\Phi}{db^2}}{\dfrac{d\Phi_0}{da^2} - \dfrac{d\Phi_0}{db^2}}$$

of ψ internally and externally, see Art. 450 above.

454.] The assumption $\sigma = \dfrac{s}{\varpi}$ is by no means arbitrary but necessitated by the condition of proportional decay or the maintenance of the current type in accordance with the results established in Chap. XXII. For if s be the specific resistance of the substance and h the thickness of the shell at any point, the condition of proportional decay with the type of currents selected is

$$\frac{F - \dfrac{d\psi}{dx}}{u} = \frac{G - \dfrac{d\psi}{dy}}{v} \quad \text{and} \quad \frac{d\psi}{dz} = 0,$$

where
$$\frac{s}{h} = C\,\frac{F - \dfrac{d\psi}{dx}}{u}, \quad C \text{ some constant.}$$

Now ψ satisfies the equations

$$\frac{d\psi}{d\nu} = lF + mG \text{ upon } S,$$

and
$$\nabla^2\psi = 0 \text{ within } S,$$

whence $\psi = Cxy$ where C is a constant, and the further condition

$$\frac{s}{h} \propto \frac{F - \dfrac{d\psi}{dx}}{u},$$

requires that $h \propto \varpi$.

455.] Hitherto we have confined our attention to cases in which given systems of currents have been supposed to be established in conducting sheets of certain forms, and then allowed to decay by the dissipation of their energy into heat under the influence of their mutual action.

We have now to consider the more general case of such conductors placed in a given magnetic field, varying from time to time according to any assigned law, and to investigate the properties of the total resultant field arising from the given magnetic field, or, as we shall generally call it, the *external field* and the field of the induced currents in the conductor, as these last decay by dissipation under the influence of their mutual inductive action and that of the external field.

As a simple example of the application of the field equations to such a problem, we will take the case of an infinite plane conducting sheet in a variable external field, and will *assume* that a system of currents, with current function of any type, has been established in the sheet by induction.

Let u, v, 0 be the component currents in the sheet at any instant, F, G, 0 the components of their vector potential, and Ω their magnetic potential.

Also, let F_0, G_0, H_0, Ω_0 be corresponding quantities arising from the given external field, these last being given functions both as to space and time, while the former are quantities to be found.

Our equations are therefore the same as those of Art. 448 above, with the substitution of $F+F_0$ and $G+G_0$ for F and G.

Therefore we have at the sheet

$$\sigma u = -\frac{dF}{dt} - \frac{dF_0}{dt} - \frac{d\psi}{dx}, \qquad \sigma v = -\frac{dG}{dt} - \frac{dG_0}{dt} - \frac{d\psi}{dy},$$

$$0 = -\frac{dH_0}{dt} - \frac{d\psi}{dz},$$

$$\frac{dF}{dz} = -2\pi u, \qquad \frac{dG}{dz} = -2\pi v,$$

$$F = \frac{dP}{dy}, \qquad G = -\frac{dP}{dx},$$

$$\therefore\ R\frac{d}{dz}\Big(\frac{dF}{dy}-\frac{dG}{dx}\Big)-\frac{d}{dt}\Big(\frac{dF}{dy}-\frac{dG}{dx}\Big)=\frac{d}{dt}\Big(\frac{dF_0}{dy}-\frac{dG_0}{dx}\Big), \qquad R=\frac{\sigma}{2\pi},$$

$$\therefore \qquad R\frac{d^2\Omega}{dz^2}-\frac{d}{dt}\cdot\frac{d\Omega}{dz}=\frac{d}{dt}\cdot\frac{d\Omega_0}{dz}, \qquad \text{(A)}$$

$$\text{or} \qquad R\frac{d\gamma}{dz}-\frac{d\gamma}{dt}=\frac{d\gamma_0}{dt},$$

γ and γ_0 being the magnetic forces at the sheet normal to the plane arising from the induced and external field respectively.

Since $\frac{d\psi}{dz}+\frac{dH_0}{dt}=0$ at the sheet, and $\nabla^2\psi=0$ in other parts of space, it follows that ψ is not generally zero, and it cannot be determined until F_0, G_0, and H_0 (assumed to be known) are actually given in terms of x, y, z, and t.

456.] The solution of the problem, therefore, involves the determination of Ω as a function of x, y, z, t satisfying the conditions Ω finite and $\nabla^2\Omega=0$ at all points of space not upon the sheet, and the equation (A) at the sheet, or when $z=0$, Ω_0 being a given function of x, y, z, and t.

For example, let the given external field be that of a unit pole moving normally to the sheet with the velocity (w).

Let a_0, b_0, c_0 be the initial coordinates of the pole and a, b, c its coordinates at any time t, then we have

$$a=a_0, \qquad b=b_0, \qquad c=c_0+wt,$$

$$\Omega_0=\frac{1}{\sqrt{(x-a_0)^2+(y-b_0)^2+(z-c_0-wt)^2}}=\frac{1}{r}, \text{ suppose.}$$

If equation (A) can be solved for all values of x, y, z, and t, the value of Ω thus found will of course satisfy (A) at the sheet, and if it also satisfies the remaining conditions it must be the required solution.

But the general solution of (A) gives

$$R\frac{d\Omega}{dz}-\frac{d\Omega}{dt}=\frac{d\Omega_0}{dt},$$

i.e. changing the variables from x, y, z, t to x, y, ζ, t where $z=\zeta-Rt$

$$-\frac{d\Omega}{dt}=\frac{d\Omega_0}{dt}+R\frac{d\Omega_0}{d\zeta},$$

$$\text{or} \qquad -\Omega=\Omega_0+R\int\frac{d\Omega_0}{d\zeta}\cdot dt,$$

where ζ is to be replaced by $z+Rt$ after the integration of the second term on the right-hand side. Hence in this case

$$-\Omega = \frac{1}{r} - \frac{R}{w+R} \cdot \frac{1}{r}$$

or

$$\Omega = \frac{-w}{w+R} \cdot \frac{1}{r}.$$

This value of Ω satisfies equation (A) everewhere and therefore upon the sheet, as well as the condition $\nabla^2\Omega = 0$, but on the positive side of the sheet it becomes infinite at the point a_0, b_0, c, i. e. at the pole, it is therefore inadmissible as the value of Ω on the positive side of the sheet.

If, however, we write for Ω_0 $\frac{1}{r'}$ or Ω_0' instead of $\frac{1}{r}$ where

$$r' = \sqrt{(x-a_0)^2 + (y-b_0)^2 + (z+c_0+wt)^2},$$

i.e. if r' be the distance of x, y, z from the optical image of the moving pole in the sheet, we observe that equation (A) on the sheet, i. e. when $z = 0$, is satisfied provided

$$R\frac{d\Omega}{dz} - \frac{d\Omega}{dt} = -\frac{d\Omega'_0}{dt} = -\frac{d}{dt} \cdot \frac{1}{r'},$$

whence, as before,

$$\Omega = \frac{1}{r'} + R\int \frac{d}{d\zeta} \cdot \frac{1}{r'}\, dt, \quad r' = \sqrt{(x-a_0)^2 + (y-b_0)^2 + (\zeta + \overline{w-R}t + c_0)^2}$$

$$= \frac{w}{w-R} \cdot \frac{1}{r'}.$$

And this value of Ω is finite at every point on the positive side of the sheet and satisfies the condition $\nabla^2\Omega = 0$.

Therefore the required value of Ω is

$$\frac{w}{w-R} \cdot \frac{1}{r'}$$

on the positive side of the sheet.

Since $\Omega = -\frac{dP}{dz}$, the general value of P satisfies the condition

$$R\frac{dP}{dz} - \frac{dP}{dt} = -\frac{d}{dt}\int \Omega'_0\, dz\,;$$

whence, reasoning as before, we should have found

$$P = -\frac{w}{w-R} \log\,(r' + z + c)$$

on the positive side of the sheet.

457.] For any practical application we must take w negative or the pole approaching the sheet from an infinite distance, whence we get on the positive side

$$P = -\frac{w}{w+R}\log(r'+z+c),$$

$$\frac{dP}{dz} = -\frac{w}{w+R}\cdot\frac{1}{r'} = -\Omega,$$

$$\therefore\quad \Omega = \frac{w}{w+R}\cdot\frac{1}{r'},\quad \text{and}\quad Z = -\frac{d\Omega}{dz} = +\frac{w}{w+R}\,\frac{z+c}{r'^3}.$$

Therefore at the pole there is a repulsive force from the disk equal to

$$\frac{1}{4c^2}\cdot\frac{w}{w+R}.$$

If the pole were not moving normally to the disk but with component velocities, u, v, w, the equation in P would become

$$-P = \log(r'+z+c)$$
$$+R\int_0^t \frac{1}{\sqrt{(x-a_0-ut)^2+(y-b_0-vt)^2+(\zeta-Rt+wt+c_0)^2}},$$

or $\quad -P = \log(r'+z+c)$

$$+\frac{R}{V}\log\left\{\frac{r'}{V}+\frac{u(a-x)+v(b-y)+\overline{w-R}\,(c+z)}{V^2}\right\},$$

where $\qquad V^2 = u^2+v^2+(w-R)^2,$

whence $\Omega = -\dfrac{dP}{dz}$ reduces to

$$\frac{(u^2+v^2+w^2-Rw)\,r'+V\{u(a-x)+v(b-y)+w(c+z)\}}{r'V\{r'V+u(a-x)+v(b-y)+(w-R)(c+z)\}}.$$

458.] Again, suppose the external magnetic field to be that of a unit pole describing a circle parallel to the sheet with uniform angular velocity (ω).

If the origin be taken at the projection of the centre of this circle on the sheet it follows that Ω_0, and therefore also Ω, are functions of the time merely so far as they are functions of the angle (ϕ suppose) between the radius rector of the projection of the point to which they refer and the radius vector of the poles projection, and that

$$dt = \frac{d\phi}{\omega}.$$

If therefore (a) be in this case the radius of the circle described by the pole, and we use the Ω_0', function as above described, we have

$$\Omega_0' = \frac{1}{\sqrt{r^2+a^2-2ra\cos\phi+(z+c)^2}} = \frac{1}{\rho}, \text{ suppose,}$$

and

$$-P = \log(\rho+z+c) + \frac{R}{\omega}\cdot\int\frac{d\phi}{\rho},$$

where $\zeta - \frac{R}{\omega}\phi$ is written for z in the expression under the integral signs before integration and $z+\frac{R}{\omega}\phi$ for ζ after integration, (r) being the distance of the projection of the point on the disk from the origin.

Whence Ω or $-\frac{dP}{dz}$ at any point can be found. The integration cannot be effected in finite terms.

459.] The question last treated may also be investigated by the application of the formulae of electromotive force in a moving conductor.

For example, let there be a plane conducting disk infinitely large revolving with uniform angular velocity ω about a normal in a given magnetic field.

If γ and γ_0, a and a_0, β and β_0 be the components of magnetic force at any point in the plane of the disk arising from the induced currents and the given magnetic field respectively, the components of the electromotive force of the motion are

$$(\gamma+\gamma_0)\frac{dy}{dt} - \frac{d\psi'}{dx}, \qquad -(\gamma+\gamma_0)\frac{dx}{dt} - \frac{d\psi'}{dy},$$

$$-(a+a_0)\frac{dy}{dt} + (\beta+\beta_0)\frac{dx}{dt} - \frac{d\psi'}{dy},$$

where

$$\psi' = F\frac{dx}{dt} + G\frac{dy}{dt} + H\frac{dz}{dt}.$$

It is usual to include the ψ' with the potential, if any, of electrical distribution in one symbol ψ, so that, if the origin be taken at the point where the axis meets the disk, in which case

$$\frac{dy}{dt} = \omega x, \quad \frac{dx}{dt} = -\omega y, \quad \frac{dz}{dt} = 0,$$

the equations become

$$\sigma u = (\gamma+\gamma_0)\,\omega x - \frac{d\psi}{dx}, \qquad \frac{dF}{dz} = -2\pi u, \quad \frac{dG}{dz} = -2\pi v;$$

$$\sigma v = (\gamma+\gamma_0)\,\omega y - \frac{d\psi}{dy},$$

$$0 = -\omega\,\{(\alpha+\alpha_0)\,x + (\beta+\beta_0)\,y\} - \frac{d\psi}{dz}.$$

Eliminating ψ from the first two and writing R for $\frac{\sigma}{2\pi}$, we get

$$R\frac{d}{dz}\left\{\frac{dF}{dy} - \frac{dG}{dx}\right\} = -\omega\left\{x\frac{d}{dy}(\gamma+\gamma_0) - y\frac{d}{dx}(\gamma+\gamma_0)\right\},$$

or

$$R\frac{d\gamma}{dz} - \omega\frac{d\gamma}{d\theta} = \omega\frac{d\gamma_0}{d\theta},$$

an equation agreeing with that of Art. 455 above.

To determine ψ we have, if ϕ be the current function and therefore

$$u = \frac{d\phi}{dy}, \quad v = -\frac{d\phi}{dx}, \quad \text{and} \quad x^2+y^2 = r^2,$$

$$\sigma\frac{d\phi}{d\theta} = (\gamma+\gamma_0)\,\omega r^2 - \frac{d\psi}{dz}, \quad \sigma r\frac{d\phi}{dr} = \frac{d\psi}{d\theta},$$

$$\frac{d\psi}{dz} = -\omega\,\{(\alpha+\alpha_0)\,x + (\beta+\beta_0)\,y\}.$$

460.] It must be carefully remembered that ψ in this case is not the potential of electrical distribution, but differs therefrom by the quantity ψ', as above explained.

For suppose the disk at rest and the field revolving, and let the H_0 of the field be zero, then, as we know, a quantity P_0 may be found from which α_0, β_0, γ_0, F_0 and G_0 may be deduced by differentiation in the same way as α, β, γ, F and G may be deduced from the P of the induced current field. Also we know in this case that if ψ be the potential of electrical distribution, we have

$$\frac{d\psi}{dz} + \frac{dH_0}{dt} = \frac{d\psi}{dz} = 0,$$

whereas the aforesaid equation gives

$$\frac{d\psi}{dz} = -\omega\left\{\left(\frac{dG}{dz} + \frac{dG_0}{dz}\right)x + \left(\frac{dF}{dz} + \frac{dF_0}{dz}\right)y\right\}$$

$$\frac{d\psi}{dz} = -\omega\frac{d}{dz}\{(F+F_0)y-(G+G_0)x\}$$
$$= \quad \frac{d}{dz}\left\{(F+F_0)\frac{dx}{dt} + (G+G_0)\frac{dy}{dt}\right\}$$
$$= \quad \frac{d\psi'}{dz}.$$

Similarly in this case

$$\frac{d\psi}{d\theta} = \sigma r\frac{d\phi}{dr} = -\frac{\sigma}{2\pi}r\frac{d^2P}{drdz} = -\omega r\frac{d}{dr}\left(\frac{dP}{d\theta}+\frac{dP_0}{d\theta}\right)$$
$$= -\omega\frac{d}{d\theta}\left\{r\frac{d}{dr}(P+P_0)\right\}$$
$$= -\frac{d}{d\theta}\left\{\omega x\frac{d}{dx}(P+P_0)+\omega y\frac{d}{dy}(P+P_0)\right\}$$
$$= \quad \frac{d}{d\theta}\left\{\frac{dy}{dt}(G+G_0)\frac{dx}{dt}(F+F_0)\right\}$$
$$= \quad \frac{d\psi'}{d\theta}.$$

Similarly for $\frac{d\psi}{dr}$.

Proving that the ψ, as determined from the above written equations, is in this case the ψ', omitted from the electromotive force of the motion and not that of electrical distribution. See above, Art. 403.

461.] In Chap. XXII we proved that every magnetic system external to a given closed surface S may be replaced by a current system upon S whose *magnetic* effects at all points within S are exactly equivalent to those of the given magnetic system, but that the *electromotive* forces arising from the S system and of the given external system are not necessarily equivalent throughout the interior of S, but may differ from each other by forces derived from a potential function.

In cases of conductors, solid or superficial, placed in any given magnetic field, we may often simplify the investigation of the inductive action by supposing the field replaced by this equivalent current system upon any properly chosen closed surface S surrounding the conductors.

This surface is generally referred to briefly as the *equivalent sheet*, and the currents thereon as the *equivalent currents*.

If, starting from this equivalent sheet and current system as own, we arrive at any results concerning the induction phenona on the given conductor surrounded by S, we conclude that same results hold true for the state of the conductor in the field the actual magnetic system, provided only a suitable additional ctrostatic charge be placed upon the conductor neutralising above-mentioned difference of electromotive force which may st between the original field and the equivalent system.

That is to say if, in the case of the conductor under the luence of the *equivalent* system, we find a certain current ction ϕ and electric potential ψ, then in the case of the conctor under the influence of the *actual* system we should have same current function ϕ, but generally an electric potential ction $\psi+\psi_0$ different from ψ.

If the conductor be a spherical surface, and the equivalent sheet oncentric spherical surface, the potential ψ is always zero, and same is true for an infinite conducting plane under the luence of an infinite parallel plane equivalent sheet.

462.] We will briefly reconsider the induction phenomena in inite conducting planes influenced by given current systems parallel plane sheets.

In this case the equations of Art. 445 above hold true with additional condition $H_0=0$.

Therefore we get $\dfrac{d\psi}{dz}=0$, and since $\nabla^2\psi=0$ at all points not the plane we have $\psi=0$ everywhere.

Therefore

$$R\frac{d}{dz}\left(\frac{dF}{dx}+\frac{dG}{dy}\right)=\frac{d}{dt}\left(\frac{dF}{dx}+\frac{dG}{dy}\right)+\frac{d}{dt}\left(\frac{dF_0}{dx}+\frac{dG_0}{dy}\right).$$

Also $\dfrac{dF_0}{dx}+\dfrac{dG_0}{dy}=0$, since the currents are closed upon the iivalent sheet, and therefore if $\dfrac{dF}{dx}+\dfrac{dG}{dy}=0$ at any instant, have

$$\frac{d}{dt}\left(\frac{dF}{dx}+\frac{dG}{dy}\right)=0,$$

d therefore $\dfrac{dF}{dx}+\dfrac{dG}{dy}=0$ always.

infinite distance from the conductor, and to be, along with its differential coefficients, continuous at all points.

Suppose P_0 to be capable of expression in a series of terms of the form
$$A_0 e^{mz} \phi(x, y) \cos kt,$$
where (since $\nabla^2 P_0 = 0$ within and in the neighbourhood of the conductor) we must have
$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + m^2\right)\phi = 0.$$

Assume for the value (P') of P within the conductor the expression
$$(\psi(z)\cos kt + \chi(z)\sin kt)\,\phi \,.\, e^{mz},$$
and for the value (P) of P without the conductor the expression
$$\{A\cos(kt) + B\sin(kt)\}\,\phi e^{-mz},$$
since P outside vanishes at infinity and satisfies $\nabla^2 P = 0$.

The equation
$$\frac{\sigma}{4\pi}\,.\,\nabla^2 P = -\frac{d}{dt}(P + P_0)$$
becomes therefore
$$\left(\frac{d^2\psi}{dz^2} + 2m\frac{d\psi}{dz}\right)\cos kt + \left(\frac{d^2\chi}{dz^2} + 2m\frac{d\chi}{dz}\right)\sin kt$$
$$= \frac{4\pi\kappa}{\sigma}\cdot\{-\chi(z)\cos kt + (A_0 + \psi(z))\sin kt\},$$
whence by equating coefficients of $\sin kt$ and $\cos kt$ we obtain two equations for the determination of ψ and χ.

475.] The general solution is somewhat complicated *, but if, as frequently happens, $\frac{4\pi\kappa}{\sigma}$ be small, so that its square and higher powers may be neglected, since ψ and χ depend at least upon the first power of $\frac{4\pi\kappa}{\sigma}$, we may neglect them when they appear on the right-hand side of the above equation as multiplied by $\frac{4\pi\kappa}{\sigma}$, whence $\psi(z) = 0$ and
$$\frac{d^2\chi}{dz^2} + 2m\frac{d\chi}{dz} = \frac{4\pi\kappa}{\sigma}A_0 = \lambda A_0, \text{ suppose,}$$
or
$$\chi = \frac{\lambda A_0}{2m}(z + C) + C' e^{-2mz}.$$

* The approximation is equivalent to neglecting the action of the induced currents in the conductor in comparison with that of the given external field.

The last term is inadmissible, because $\chi(z)e^{mz}$ would otherwise become infinitely great when $-z$ is infinitely large.

Hence within the conductor the value of P is

$$\frac{\lambda A_0}{2m}(z+C)e^{mz}\phi\sin kt.$$

In the external field it is

$$Ae^{-mz}\phi\sin kt,$$

where

$$\frac{\lambda A_0}{2m}C=A,\quad \frac{\lambda A_0}{2m}+\frac{\lambda A_0}{2}C=-mA=-\frac{\lambda A_0}{2}C,$$

from the condition of continuity of P and $\dfrac{dP}{dz}$;

$$\therefore\quad C=-\frac{1}{2m},\qquad A=-\frac{\lambda A_0}{4m^2},$$

and the internal value of P or P' is

$$\frac{\lambda A_0}{2m}\left\{z-\frac{1}{2m}\right\}e^{mz}\phi\sin kt.$$

And the external value of P is

$$-\frac{\lambda A_0}{4m^2}e^{-mz}\phi\sin kt.$$

476.] If the conductor be a sphere of radius a, let the equivalent current sheet be the surface of a concentric sphere, then we know that a magnitude P_0 may be found such that

$$F_0=\left(z\frac{d}{dy}-y\frac{d}{dz}\right)P_0,\quad G_0=\left(x\frac{d}{dz}-z\frac{d}{dx}\right)P_0,$$
$$H_0=\left(y\frac{d}{dx}-x\frac{d}{dy}\right)P_0;$$

and therefore all the required equations can be satisfied by taking

$$F=\left(z\frac{d}{dy}-y\frac{d}{dz}\right)P,\quad G=\left(x\frac{d}{dz}-z\frac{d}{dx}\right)P,$$
$$H=\left(y\frac{d}{dx}-x\frac{d}{dy}\right)P,\quad \psi=0,$$

provided P satisfies the equation

$$\frac{\sigma}{2\pi}\nabla^2P=-\frac{d}{dt}(P+P_0);$$

and be finite everywhere and continuous, as well as its differential coefficients, and in the external space satisfy the equation

$$\nabla^2 P = 0.$$

I_0 also satisfies the equation

$$\nabla^2 P_0 = 0,$$

and therefore may be expressed in a series of spherical harmonics with coefficients functions of the time.

Let any term in P_0 be of the form

$$A_0 Y_n \left(\frac{r}{a}\right)^n \cos(\kappa t),$$

and let the corresponding terms in P be

$$\psi(r)\left(\frac{r}{a}\right)^n Y_n \cos(\kappa t) + \chi(r)\left(\frac{r}{a}\right)^n Y_n \sin \kappa t$$

for the internal space, and

$$A\left(\frac{a}{r}\right)^{n+1} Y_n \cos \kappa t + B\left(\frac{a}{r}\right)^{n+1} Y_n \sin \kappa t$$

for the external space, then the equation

$$\frac{\sigma}{4\pi}\nabla^2 P = -\frac{d}{dt}(P+P_0)$$

within the sphere becomes

$$\left(\frac{d^2\psi}{dr^2} + \frac{2}{r}\cdot\frac{d\psi}{dr} - \frac{n(n+1)}{r^2}\psi\right)\cos \kappa t$$
$$+\left(\frac{d^2\chi}{dr^2} + \frac{2}{r}\cdot\frac{d\chi}{dr} - \frac{n(n+1)}{r^2}\chi\right)\sin \kappa t$$
$$= -\frac{4\pi\kappa}{\sigma}\{\chi(r)\cos \kappa t - (A_0 + \psi r)\sin \kappa t\}.$$

477.] If $\frac{4\pi\kappa}{\sigma}$ be small, and its squares and higher powers be neglected, the equation is satisfied by neglecting the term

$$\psi(r)\left(\frac{r}{a}\right)^n Y_n \cos \kappa t$$

in the assumed expression for P internally, and determining χ from the equation

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} - \frac{n(n+1)}{r^2}\chi = \frac{4\pi\kappa}{\sigma}A_0.$$

plane at the angle $\tan^{-1}\frac{R}{u}$ and magnetised parallel to the plane sheet.

When the motion of the given pole is inclined to the plane sheet the general result is the same, but the inclination of the magnetised bar is different. When the pole describes a circle parallel to the sheet the magnetic effect of the induced currents is that of a helix on the right cylinder whose transverse section is the described circle, magnetised in the direction of a tangent to this cylinder perpendicular to the axis inclined at the angle $\tan^{-1}\frac{R}{u}$ to the plane, and terminating in the instantaneous optical image of the pole.

The whole investigation is given with much instructive detail by Messrs. Mascart and Joubert in their treatise already quoted, and the results arrived at are identical with those obtained by the preceding analytical treatment.

465.] In the last chapter we investigated the case of the spherical conducting sheet in any field.

If the field be replaced by the equivalent system on a concentric spherical surface, the equations of Art. 451 hold with the substitution of $P+P_0$ for P.

Hence, as in that Article, we have $\psi=0$.

Also $$R\left(\frac{dP}{dr}-\frac{dP'}{dr}\right)=\frac{d}{dt}\cdot(P+P_0) \text{ at the sheet.}$$

P_0 at any given instant must be expressible in a series of spherical harmonics, and therefore at any time t must be of the form $\Sigma A_0\left(\frac{r}{a}\right)^n Y_n$, where A_0 is a given function of t.

Whence, if A_0 be of the form $A_0\cos(\kappa t+a)$ where A_0 is constant, we get, as in the Article referred to, P and P' being external and internal values of P respectively,

$$P=-A_0\cos\beta\cos(kt+a-\beta)Y_n\left(\frac{a}{r}\right)^{n+1}, \text{ if } \tan\beta=\frac{(2n+1)R}{ak},$$

$$P'=-A_0\cos\beta\cos(kt+a-\beta)Y_n\left(\frac{r}{a}\right)^n.$$

$$\phi=-\frac{2n+1}{4\pi}\frac{A_0}{a}\cos\beta\cos(kt+a-\beta)Y_n.$$

466.] For the particular case of a uniform field of force $F\cos(kt+a)$ parallel to z, we have ϕ of the form Cz.

In this case therefore, as shown above (Art. 446), the currents are in circles parallel to the plane of x, y, and by what was proved in the article referred to, the interior of the shell is a field of uniform force parallel to z, and the external field is that of a simple magnet at the centre of the sphere with axis in the axis of z and moment at any time t equal to

$$\tfrac{1}{3} Fa^3 \cos\beta \cos(kt+a-\beta) \quad \text{where} \quad \tan\beta = \frac{3R}{ak}.$$

467.] The case of a conducting spherical sheet of radius (a), revolving with constant angular velocity (ω) about a diameter, coinciding with the axis of z in a uniform magnetic field leads to exactly similar treatment.

For let the sheet be at rest and the field revolve round the same axis with the same angular velocity reversed, the relative motion is the same.

With the same notation as before, we get at the surface of the sheet the equation

$$R\left(\frac{dP}{dr} - \frac{dP'}{dr}\right) = \frac{d}{dt}\cdot(P+P_0).$$

If ϕ be the azimuthal angle between a point fixed with reference to the revolving field and a point in space, we have $\dfrac{d\phi}{dt} = \omega$, and therefore

$$R\left(\frac{dP}{dr} - \frac{dP'}{dr}\right) = \omega\frac{d}{d\phi}(P+P_0).$$

If P_0 be expressed in a series of spherical harmonics, the most general form of that of the n^{th} order is, as we know, a series of terms of the form

$$A_0 \psi_n^k(\theta)\cos(k\phi + a)\left(\frac{r}{a}\right)^n.$$

If we assume for P a similar expression, the term in P of the n^{th} order and type, k will be

$$\psi_n^k(\theta)\chi(\phi)\left(\frac{a}{r}\right)^{n+1},$$

and that of P' will be

$$\psi_n^k(\theta)\chi(\phi)\left(\frac{r}{a}\right)^n,$$

and the above written equation becomes

$$-\frac{(2n+1)}{a} R\chi(\phi) - \omega \frac{d\chi}{d\phi} = -A_0 k\omega \sin(k\phi + a),$$

or writing p for $\frac{(2n+1)R}{a\omega k}$,

$$\frac{d\chi}{d\phi} + pk\chi = kA_0 \sin(k\phi + a).$$

And therefore, as in the last case,

$$\chi = -A_0 \cos\beta \cos(k\phi + a - \beta),$$

where $\tan\beta = p$.

If the given field be one of uniform force F, parallel to the axis of revolution, P_0 assumes the form $A_0 \cos\theta$, and is independent of ϕ. In this case therefore P is zero, and there is no induction of currents in the sheet.

If the uniform force F be perpendicular to the axis of revolution, P_0 assumes the form

$$A_0 \sin\theta \cos(\phi + a),$$

and P becomes

$$-A_0 \cos\beta \sin\theta \cos(\phi + a - \beta).$$

Within the revolving sheet therefore the field is that of a uniform force perpendicular to the axis of revolution, and inclined at the angle β to the original line and with intensity diminished in the ratio of $\cos\beta$ to unity, where

$$\tan\beta = \frac{3R}{a\omega}.$$

The induced currents on this sphere are the same as if the sphere were fixed in a field with uniform force $F\cos\beta$ in the direction aforesaid.

By what has been already proved therefore the internal field of the induced currents is one of uniform force, and the external field is that of a small magnet at the centre of the sheet, with axis parallel to the aforesaid direction and moment equal to

$$\tfrac{1}{2} Fa^3 \cos\beta.$$

In the particular case of the spherical shell revolving round the axis of z with uniform angular velocity (ω) in a field of uniform force a_0, β_0, γ_0, we may also proceed as follows.

468.] We know, Art. 446, that the system of currents derived from the current function

$$\phi = (Ax+By+Cz)a,$$

i. e. with components,

$$Bz-Cy, \quad Cx-Az, \quad Ay-Bx,$$

produces upon and within the spherical surface a field of uniform force, whose components α, β, γ are

$$\frac{8\pi a}{3}\cdot A, \quad \frac{8\pi a}{3}\cdot B, \quad \text{and} \quad \frac{8\pi a}{3}\cdot C.$$

The equations of Ohm's law are therefore with such a system of currents on the shell,

$$\frac{3\sigma}{8\pi a}(\beta z-\gamma y) = (\gamma+\gamma_0)\omega x - \frac{d\psi}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{3\sigma}{8\pi a}(\gamma x-\alpha z) = (\gamma+\gamma_0)\omega y - \frac{d\psi}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{3\sigma}{8\pi a}(\alpha y-\beta x) = -\omega\{(\alpha+\alpha_0)x+(\beta+\beta_0)y\} - \frac{d\psi}{dz}. \quad . \quad (3)$$

Eliminating ψ between the first two equations, we get

$$\frac{3\sigma}{4\pi a}\gamma = 0.$$

Showing that γ and therefore C is zero, and that the current function is reduced to

$$Ax+By \quad \text{or} \quad \frac{3}{8\pi a}\cdot(\alpha x+\beta y).$$

The elimination of ψ between 1 and 3, 2 and 3, gives

$$\frac{3\sigma}{4\pi a}\beta = \omega(\alpha+\alpha_0), \quad \text{and} \quad -\frac{3\sigma}{4\pi a}\alpha = \omega(\beta+\beta_0),$$

whence, writing p for $-\dfrac{3\sigma}{4\pi a\omega}$, we get

$$\alpha^2+\beta^2 = \frac{\alpha_0^2+\beta_0^2}{1+p^2}, \qquad \frac{\beta}{\alpha} = \frac{\beta_0+p\alpha_0}{\alpha_0-p\beta_0}.$$

If $\qquad \dfrac{\beta_0}{\alpha_0} = \tan\epsilon, \quad \dfrac{\beta}{\alpha} = \tan\theta, \quad$ and $\quad p = \tan\delta,$

$$\sqrt{\alpha^2+\beta^2} = \sqrt{\alpha_0^2+\beta_0^2}\cos\delta, \qquad \tan\theta = \tan(\epsilon+\delta).$$

Or the system of induced currents on the moving sphere is the same as that which on the fixed sphere would produce a constant force in a line perpendicular to the axis of rotation inclined at the constant angle $\tan^{-1}\frac{3R}{a\omega}$ to the direction of the force of the given field resolved perpendicular to the same axis, and whose intensity is equal to the resolved part of the force of the field in the direction of that line.

469.] Next, suppose that the sphere in the last example is replaced by the spheroid

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1,$$

rotating round the axis of figure c, and let the thickness of the shell be proportional at each point to the perpendicular from the centre on the tangent plane.

If we assume a system of currents to exist upon this shell, such that the current function ϕ is

$$-(Ax+By),$$

the components currents will be

$$-\varpi\frac{Bz}{c^2},\quad \varpi\frac{Az}{c^2},\quad \frac{\varpi}{a^2}(Bx-Ay).$$

And these will produce within and upon the sheet the uniform force, whose components are

$$-4\pi a^2c\left\{\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{dc^2}\right\}A,\quad -4\pi a^2c\left\{\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{dc^2}\right\}B,$$

and zero respectively.

Denote these force components by α and β, so that

$$u=\frac{\frac{\varpi z}{c^2}\beta}{4\pi a^2c\left\{\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{dc^2}\right\}},\qquad v=-\frac{\frac{\pi z}{c^2}\alpha}{4\pi a^2c\left\{\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{dc^2}\right\}},$$

$$\text{and}\qquad w=-\frac{\frac{\varpi}{a^2}}{4\pi a^2c\left\{\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{dc^2}\right\}}\{\beta x-\alpha y\}.$$

If now for σ we write $\frac{s}{\varpi}$, and for $-\frac{1}{4\pi a^2 c\left\{\frac{d\Phi_0}{da^2}+\frac{d\Phi_0}{dc^2}\right\}}$ we write p, our Ohm's law equations become

$$-sp\frac{\beta z}{c^2} = \omega\gamma_0 x - \frac{d\psi}{dx},$$

$$sp\frac{\alpha z}{c^2} = \omega\gamma_0 y - \frac{d\psi}{dy},$$

$$\frac{sp}{a^2}(\beta x - \alpha y) = -\omega\left\{(\alpha+\alpha_0)x + (\beta+\beta_0)y\right\} - \frac{d\psi}{dz}.$$

Therefore $$-sp\beta\left(\frac{1}{c^2}+\frac{1}{a^2}\right) = \omega(\alpha+\alpha_0),$$

$$sp\alpha\left(\frac{1}{c^2}+\frac{1}{a^2}\right) = \omega(\beta+\beta_0),$$

or $$-\frac{sp}{\omega}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)\beta = \alpha+\alpha_0,$$

$$\frac{sp}{\omega}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)\alpha = \beta+\beta_0;$$

therefore $$p_1\beta+\alpha = -\alpha_0, \quad p_1\alpha-\beta = \beta_0,$$

where $$p_1 = \frac{ps}{\omega}\left(\frac{1}{a^2}+\frac{1}{c^2}\right),$$

whence we arrive at similar conclusions to those in the case of the sphere.

470.] In the case of the revolving sphere of Article 468 the quantity ψ is identical with ψ' or $F\dot{x}+G\dot{y}+H\dot{z}$.

For we get from equations 1, 2, 3 of that Article

$$\psi = \frac{\omega\gamma_0}{2}(x^2+y^2) + \frac{3\sigma}{8\pi a}z(\alpha y - \beta x).$$

Also in this case

$$F = -\gamma_0\frac{y}{2} + (\beta+\beta_0)\frac{z}{2}, \qquad G = \gamma_0\frac{x}{2} - (\alpha+\alpha_0)\frac{y}{2},$$

$$H = (\alpha+\alpha_0)\frac{y}{2} - (\beta+\beta_0)\frac{x}{2};$$

$$\dot{x} = -\omega y, \qquad \dot{y} = \omega x, \qquad \dot{z} = 0.$$

And therefore, since

$$\omega(\alpha+\alpha_0) = \frac{3\sigma}{4\pi a}\beta, \quad \text{and} \quad \omega(\beta+\beta_0) = -\frac{3\sigma}{4\pi a}\cdot\alpha,$$

$F\dot{x}+G\dot{y}+H\dot{z}$, or ψ', reduces to

$$\frac{\omega\gamma_0}{2}(x^2+y^2)+\frac{3\sigma}{8\pi a}z(ay-\beta x),$$

i.e. to ψ, whence we know that there is no distribution of free electricity in this case.

In the case of the revolving spheroid of the present Article, $F\dot{x}+G\dot{y}+H\dot{z}$ or ψ' will be found by similar treatment to differ from ψ by a quantity of the form

$$sp\left(\frac{1}{c^2}-\frac{1}{a^2}\right)z(ay-\beta x).$$

Indicating an electrical distribution with potential of the form $Mzy-Nzx$ upon the spheroid.

And therefore with values $Mzy-Nzx$ within, and

$$Mzy\frac{\dfrac{d\Phi}{db^2}-\dfrac{d\Phi}{dc^2}}{\dfrac{d\Phi_0}{db^2}-\dfrac{d\Phi_0}{dc^2}}-Nzx\cdot\frac{\dfrac{d\Phi}{da^2}-\dfrac{d\Phi}{dc^2}}{\dfrac{d\Phi_0}{da^2}-\dfrac{d\Phi_0}{dc^2}}$$

without the spheroid, and of superficial density of the form

$$M'zy-N'zx$$

upon the spheroid.

471.] Hitherto we have applied the general field equations to the investigation of the phenomena of induction in closed conducting sheets of special forms situated in a variable magnetic field. The same general principles hold good in whatever be the forms of the sheets, but except in special cases their application presents very great analytical difficulties. As any closed currents are generated in the external field, a system of closed currents constituting the magnetic screen to the external field comes into existence, by induction, in the sheet.

The effect of the finite resistance of the conducting sheet is to cause these induced currents to decay by dissipation of their energy into heat. In this process of decay they vary, and thus exercise mutual inductive influences. In certain special systems, which may with propriety be called self-inductive, their intensity at any time t from their first establishment diminishes according to the $e^{-\lambda t}$ law, where λ is a coefficient depending upon the

shape and resistance of the conductor, and these special systems alone are easily amenable to mathematical treatment.

As shewn above, Art. 431, in the very important class of cases in which the external system varies periodically, the field in the interior of the sheet also varies periodically but with retarded phase and generally with diminished intensity.

472.] When we pass to the consideration of a solid conductor of any form the same general principles hold, but their application becomes very complicated.

Thus, the first effect of the excitation of the external system may be regarded as the establishment of superficial currents, constituting a perfect screen as before, and if the resistance were evanescent this screen would be always maintained and the currents would always be on the surface. But the effect of the resistance is to impair this screening influence, so that if the external system remained unchanged the superficial currents would vary by resistance, and thus by their variation modify their own intensities and induee currents in the interior mass, which would again modify the superficial currents. If the external system also varied the problem would be still more complicated.

The problem of induction currents, therefore, in a solid conductor in a varying magnetic field is one of great analytical difficulty even in its simplest cases, as, for instance, where the conductor is bounded by infinite parallel planes or concentric spherical surfaces. For these cases it has been treated with great generality in special papers to which the reader is referred *.

473.] By way of illustrating the general treatment, we proceed to investigate the question under very restricted and special conditions.

Consider a solid conductor of any form situated in a given varying magnetic field.

At all points within the conductor

$$\nabla^2 F = -4\pi u, \quad \nabla^2 G = -4\pi v, \quad \nabla^2 H = -4\pi w,$$

* See, amongst others, a paper by Professor C. Niven in the *Phil. Trans.* of the Royal Society, 1881, part II; also a paper by Professor H. Lamb, *Phil. Trans.*, R. S., 1883, part II; and a paper in the Philosophical Magazine, already quoted in the text, by Dr. Larmor, January, 1884.

and at all external points $\nabla^2 F$, $\nabla^2 G$, $\nabla^2 H$ are severally zero, F, G, H are everywhere finite and vanish at infinity.

F and F_0, G and G_0, H and H_0 are continuous everywhere, as well as their differental coefficients.

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dx}=0, \text{ everywhere.}$$

The component currents satisfy the conditions

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$$

at all points, and the additional condition

$$lu+mv+nw=0$$

at the surface of the condutor.

The equations of Ohm's law,

$$\sigma u=-\frac{d}{dt}(F+F_0)-\frac{d\psi}{dx}, \quad \sigma v=-\frac{d}{dt}(G+G_0)-\frac{d\psi}{dy},$$

$$\sigma w=-\frac{d}{dt}(H+H_0)-\frac{d\psi}{dz},$$

hold always at all points in the conductor.

474.] Take the case of a solid conductor bounded by an infinite plane, that of x, y suppose, and extending indefinitely towards the negative axis of z, and suppose there is a varying magnetic field in front of it.

Replace this field by the magnetically equivalent current system on a plane parallel to the boundary of the conductor as above explained.

At any point in the field of the conductor a function P_0 may be found, such that

$$F_0=\frac{dP_0}{dy}, \quad G_0=-\frac{dP_0}{dx}, \quad H_0=0.$$

And then, as we know, all the requisite conditions can be satisfied by

$$F=\frac{dP}{dy}, \quad G=-\frac{dP}{dx}, \quad H=0, \quad \psi=0,$$

provided P be so chosen as to satisfy the equation

$$\frac{\sigma}{4\pi}.\nabla^2 P=-\frac{d}{dt}(P+P_0)$$

the conductor, to be everywhere finite, to vanish at an

CHAPTER XXIV.

AMPÈRE'S AND OTHER THEORIES.

ARTICLE 484.] IN the preceding chapters we have found that two closed circuits with currents i and i' possess energy of their mutual action $ii'\iint \frac{\cos \epsilon}{r}\, ds\, ds'$, taken round both circuits in the direction of the currents, and the energy of any field of closed currents is

$$\tfrac{1}{2}\iiint (Fu + Gv + Hw)\, dx\, dy\, dz.$$

The proof of this rests ultimately on experiments made with closed conducting circuits, where no account is taken of the variation of the statical distributions of electricity or the statical potential. It is only for circuits of this character that we are strictly justified in using the above expression for the energy. We may call a system of such circuits a *purely magnetic system*.

485.] Generally in any field of currents we have what used to be called unclosed currents, that is, statical distributions forming on the surfaces of conductors, and variations of the statical potential.

According to Maxwell's theory the circuits are nevertheless all closed, if we take into account the displacement currents in the insulating or partially insulating space, and the energy of any field of currents is still represented by

$$\tfrac{1}{2}\iiint (Fu + Gv + Hw)\, dx\, dy\, dz,$$

u, v, and w including the displacement as well as the conduction currents.

Maxwell's theory, as thus extended, is consistent with experiments. It is possible however to explain experiments with closed circuits on other hypotheses concerning the laws of force or of energy between elementary currents. Ampère's law especially, as extended by Weber, has met with very general accept-

ance. And Helmholtz' treatment requires explanation with a view to the electromagnetic theory of light.

We therefore propose to devote the present chapter to the consideration of some of these hypotheses. We shall assume that every pair of current elements exert on each other a certain force, or possess energy of their separate action.

When we speak of a force acting between two current elements, we must be understood as meaning a force acting between the elementary conductors in which the currents flow in virtue of those currents; for we cannot conceive electric currents as in any other sense the subject of mechanical action. But for brevity we shall follow the example of other writers on the subject by speaking of the action as between the currents.

486.] We shall employ the following notation. If CP or ds, $C'P'$ or ds', two infinitely short lines, represent the directions of two elementary electric currents, then in our notation

$$CC' = r,$$
$$\angle PCC' = \theta,$$
$$\angle P'C'C = \theta',$$

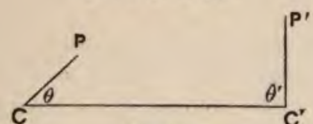

Fig. 47.

and the angle between CP and $C'P' = \epsilon$. Evidently with this notation

$$\cos\theta = -\frac{dr}{ds},$$

$$\cos\theta' = -\frac{dr}{ds'},$$

$$\cos\epsilon = \frac{d}{ds}(r\cos\theta') = -\frac{dr}{ds}\frac{dr}{ds'} - r\frac{d^2r}{ds\,ds'},$$

and we shall denote by i the current in ds, that is the quantity of electricity which passes in unit of time through a section of ds. Similarly i' shall denote the current in ds'. In the ordinary notation u, v, w are component currents per unit area of the section, so that $i\,ds$ corresponds to $u\,dy\,dz\,dx$.

If ds be an element of a closed circuit, ds' any elementary straight line, then

$$\int \frac{\cos\theta\cos\theta'}{r}\,ds = -\int \frac{\cos\epsilon}{r}\,ds.$$

For
$$\int \frac{\cos\theta\cos\theta'}{r}\,ds = \int \frac{\cos\theta\, r\cos\theta'}{r^2}\,ds$$
$$= -\int \frac{1}{r^2}\frac{dr}{ds}\,r\cos\theta'\,ds$$
$$= -\int \frac{1}{r}\frac{d}{ds}(r\cos\theta')\,ds$$

by integration by parts, the integrated term disappearing for the closed circuit

$$= -\int \frac{\cos\epsilon}{r}\,ds'.$$

487.] The four quantities r, ϵ, θ, θ' completely define the relative position of any two elements of electric currents. If therefore these elements possess energy of their mutual action, or exert a force on one another, this energy, or force, must be capable of expression as a function of r, ϵ, θ, and θ', together with $i\,ds$ and $i'ds'$.

488.] It is assumed generally in these investigations:

I. That the effect of any element of a current on any other element is directly as the product of the strengths of the currents and the lengths of the elements. That is, it is proportional to $i\,ds\,i'ds'$.

II. That every elementary current may be replaced by its components.

That is, if $i\,CP$, $i\,CQ$ be two elementary currents, both starting from C, and if CR be the diagonal of the parallelogram $RPCQ$, the two currents $i\,CP$, $i\,CQ$ are for all purposes equivalent to the single current $i\,CR$.

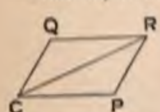

Fig. 48.

This we shall call the *law of composition.*

It is found by experiment that the effect of a sinuous current CPR can, by diminishing the dimensions of the currents, be made to differ as little as we please from that of the straight current CR. Hence we infer the truth of the law.

If ds be the length of the elementary current at C, ds_1 and ds_2 its components, we shall denote by ϵ, ϵ_1, and ϵ_2 the angles which they respectively make with ds'. And in like manner the suffixes $_1$ or $_2$ applied to any function of $r, \epsilon, \theta, \theta', ds$, and ds' shall denote that ds_1 or ds_2 is concerned in its formation.

489.] Definition.—If $f(r, \epsilon, \theta, \theta')\, ds\, ds'$, or shortly f, be such a function that for any whatever two components of ds, as ds_1 and ds_2,

$$fds = f_1 ds_1 + f_2 ds_2,$$

f is said to obey the *law of composition*. We can now prove the following proposition.

If $f(r, \epsilon, \theta, \theta')$, or shortly f, be any function which obeys the law of composition, and is symmetrical with regard to θ and θ', f must be of the form

$$\phi(r)\cos\epsilon + \psi(r)\cos\theta\cos\theta',$$

where $\phi(r)$ and $\psi(r)$ are undetermined functions of r.

By hypothesis

$$fds = f_1 ds_1 + f_2 ds_2,$$

$$\text{or} \qquad f = f_1 \frac{ds_1}{ds} + f_2 \frac{ds_2}{ds}. \qquad \qquad (1)$$

By projecting on ds',

$$\cos\epsilon\, ds_1 = \cos\epsilon_1 ds_1 + \cos\epsilon_2 ds_2,$$

$$\text{or} \qquad \cos\epsilon = \cos\epsilon_1 \frac{ds_1}{ds} + \cos\epsilon_2 \frac{ds_2}{ds}. \qquad \qquad (2)$$

Now let ϵ vary, θ and θ' remaining constant. That is, conceive ds' to be the radius vector of a cone of which C' is vertex, and r the axis. Then as ds' changes its position on the cone, ϵ changes, and ϵ_1, ϵ_2 change with it, but r, θ, and θ' are unaffected. Then by differentiation from (1) and (2),

$$\frac{df}{d\epsilon} = \frac{df_1}{d\epsilon_1}\frac{d\epsilon_1}{d\epsilon}\frac{ds_1}{ds} + \frac{df_2}{d\epsilon_2}\frac{d\epsilon_2}{d\epsilon}\frac{ds_2}{ds},$$

$$\sin\epsilon = \sin\epsilon_1 \frac{d\epsilon_1}{d\epsilon}\frac{ds_1}{ds} + \sin\epsilon_2 \frac{d\epsilon_2}{d\epsilon}\frac{ds_2}{ds};$$

and these equations being true for any three directions in any plane, and whatever the lengths of ds_1 and ds_2, we must have

$$\frac{1}{\sin\epsilon}\frac{df}{d\epsilon} = \frac{1}{\sin\epsilon_1}\frac{df_1}{d\epsilon_1} = \frac{1}{\sin\epsilon_2}\frac{df_2}{d\epsilon_2},$$

a parallelogram, one side of which coincides with the perpendicular to the plane of the circle through its centre. It is found that a current sent through the second closed circuit does not tend to move the circular conductor round its axis. But by symmetry this must be true for any law of force which has a potential. The experiment cannot therefore be relied upon as establishing Ampère's law. Again, in the experiment described in Maxwell, 2nd edition, Vol. II, § 507, modified in § 687, we have two cups of mercury on a plane, and a wire passes through them, and is bent between the cups in the form of a circular arc. The wire between the cups forms part of a voltaic circuit, the current entering through the mercury in one cup and leaving through the mercury in the other cup. If any other closed circuit be brought into the neighbourhood, it is found not to move the wire round an axis through the centre of the circle of which it forms part. In this case any movement of the wire round the axis would not alter the position of the current, but would merely place a different portion of the wire in position to carry the same current. It cannot therefore alter the potential of the electrodynamic forces if they have one.

Concerning Weber's Hypothesis.

496.] Weber gives a physical explanation of Ampère's results as follows. He assumes that two quantities of electricity, or, as we may say, electrical masses, e and e', have, in addition to their statical potential $\frac{ee'}{r}$, also a potential due to their relative motion equal to

$$-ee'\frac{C}{4r}\left(\frac{dr}{dt}\right)^2,$$

where C is a constant. This gives a repulsive force in r equal to

$$-ee'\frac{C}{4r^2}\left(\frac{dr}{dt}\right)^2+ee'\frac{C}{2r}\frac{d^2r}{dt^2},$$

or as we may write it

$$C\frac{ee'}{\sqrt{r}}\frac{d^2\sqrt{r}}{dt^2}.$$

Let us now assume that an electric current consists of equal quantities of positive and negative electricity moving with equal

velocities in opposite directions. Then considering e in ds and e' in ds', and r the distance between them, we have

$$\frac{dr}{dt} = v\frac{dr}{ds} + v'\frac{dr}{ds'},$$

v and v' being the velocities of e and e' respectively, and if v and v' be constant,

$$\frac{d^2 r}{dt^2} = v^2\frac{d^2 r}{ds^2} + v'^2\frac{d^2 r}{ds'^2} + 2vv'\frac{d^2 r}{ds\,ds'};$$

and therefore the force between e and e' is, omitting C,

$$\frac{ee'}{2r}\left\{v^2\frac{d^2 r}{ds^2} + v'^2\frac{d^2 r}{ds'^2} + 2vv'\frac{d^2 r}{ds\,ds'}\right\},$$

$$-\frac{ee'}{4r^2}\left\{v^2\left(\frac{dr}{ds}\right)^2 + v'^2\left(\frac{dr}{ds'}\right)^2 + 2vv'\frac{dr}{ds}\frac{dr}{ds'}\right).$$

The force between e in ds and $-e'$ moving with velocity $-v'$ in ds' is found from the above by changing the sign of e' and v'. It is therefore

$$-\frac{ee'}{2r}\left\{v^2\frac{d^2 r}{ds^2} + v'^2\frac{d^2 r}{ds'^2} - 2vv'\frac{d^2 r}{ds\,ds'}\right\}$$

$$+\frac{ee'}{4r^2}\left\{v^2\left(\frac{dr}{ds}\right)^2 + v'^2\left(\frac{dr}{ds'}\right)^2 - 2vv'\frac{dr}{ds}\frac{dr}{ds'}\right\}.$$

The force between e in ds and the system of e' and $-e'$ in ds' is found by adding together the two expressions. It is therefore

$$\frac{ee'}{r}2vv'\frac{d^2 r}{ds\,ds'} - \frac{ee'}{2r^2}2vv'\frac{dr}{ds}\frac{dr}{ds'}.$$

By symmetry, the force upon $-e$ in the element ds moving with velocity $-v$ is the same as the preceding. And writing i for ev and i' for $e'v'$, we find for the force between the two current elements $i\,ds$ and $i'ds'$

$$\frac{4ii'}{r}\frac{d^2 r}{ds\,ds'} - \frac{2ii'}{r^2}\frac{dr}{ds}\frac{dr}{ds'},$$

that is,

$$\frac{2ii'}{r^2}\left\{2r\frac{d^2 r}{ds\,ds'} - \frac{dr}{ds}\frac{dr}{ds'}\right\},$$

which differs only by a constant factor from Ampère's force.

497.] We have assumed v and v' to be constant. Let us now suppose v', the velocity of e' in ds' and of $-e'$ in the reverse direction, to vary with the time. Then, as before,

$$\frac{dr}{dt} = v\frac{dr}{ds} + v'\frac{dr}{ds'}.$$

But $$\frac{d^2r}{dt^2} = v^2\left(\frac{dr}{ds}\right)^2 + v'^2\left(\frac{dr}{ds'}\right)^2 + 2vv'\frac{d^2r}{ds\,ds'} + \frac{dv'}{dt}\frac{dr}{ds'}.$$

And therefore the force exerted by $+e'$ on e contains, in addition to the expression above found for it, the term

$$\frac{ee'}{2r}\frac{dv'}{dt}\frac{dr}{ds'}ds',$$

that is, $$-\frac{ee'}{2r}\frac{dv'}{dt}\cos\theta'\,ds'.$$

But from $-e'$ and $-\frac{dv'}{dt}$ will be derived the same term; so that the force on e due to the change in the velocities of e' and $-e'$ will be a repulsive force

$$-\frac{ee'}{r}\frac{dv'}{dt}\cos\theta'\,ds' \text{ in direction } r;$$

and therefore $$\frac{ee'}{r}\frac{dv'}{dt}\cos\theta\cos\theta'\,ds' \text{ in direction } ds.$$

Similarly the force on $-e$ in direction ds reversed, due to the change in velocity of e' and $-e'$, will be

$$\frac{ee'}{r}\frac{dv'}{dt}\cos\theta\cos\theta'\,ds'.$$

But these two equal forces, on e in direction ds and on e' in the reverse direction, constitute *the Electromotive force* in ds due to the time variation of v', that is of the current in ds'.

The electromotive force in ds due to a closed circuit of which ds' forms part is $$e'\frac{dv'}{dt}\int\frac{\cos\theta\cos\theta'}{r}ds',$$

that is $$\frac{di'}{dt}\int\frac{\cos\theta\cos\theta'}{r}ds',$$

that is, as shown above Art. 486,

$$-\frac{di'}{dt}\int\frac{\cos\epsilon}{r}ds'.$$

That is $-\frac{dF}{dt}$, if F denote the resultant in direction ds of the vector potential of the closed circuit. This agrees with the

laws of induction by variation of the primary circuit above obtained, Chap. XIX.

For further investigations of this kind the reader may consult among others the works mentioned in the foot-note.*

498.] It will be sufficient for our purpose here to follow the more general method of Helmholtz, in order to investigate the effect of unclosed electric currents, if such exist in the field.

The energy of two elementary currents $i\,ds$ and $i'ds'$ may be assumed to be symmetrical with regard to θ and θ'. It must therefore be of the form

$$\phi(r)\cos\epsilon + \psi(r)\cos\theta\cos\theta'.$$

The experimental law III shows, as before, that $\phi(r)$ and $\psi(r)$ are of the form $\frac{a}{r}$ and $\frac{b}{r}$ respectively, where a and b are constants.

If the currents be all in closed circuits, this can be reduced to one term involving $\frac{\cos\epsilon}{r}$. For, for any closed circuit in relation to an element ds' of another circuit,

$$ds'\int\frac{\cos\theta\cos\theta'}{r}\,ds = -ds'\int\frac{\cos\epsilon}{r}\,ds, \text{ as above shown, Art. 486.}$$

And therefore the assumed energy when applied to closed circuits is reduced to $\frac{a-b}{2}\iint\frac{\cos\epsilon}{r}\,i\,ds\,i'ds'$.

The existence of the term involving $\frac{\cos\theta\cos\theta'}{r}$ in the expression for the energy is matter of indifference so far as closed circuits are concerned.

In any case we require only to know the ratio of the constants a and b. We may therefore put the energy in the form adopted by Helmholtz,

$$T = A^2\left\{\frac{1+\kappa}{2}\frac{\cos\epsilon}{r} - \frac{1-\kappa}{2}\frac{\cos\theta\cos\theta'}{r}\right\}i\,ds\,i'ds'$$
$$= A^2\left\{\frac{1}{r}\cos\epsilon + \frac{1-\kappa}{2}\frac{d^2r}{ds\,ds'}\right\}i\,ds\,i'ds',$$

* Stefan, *Sitzungsberichte*, Vienna 1869.
Carl Neumann, *Ueber die den Kräften Electrodynamischen Ursprungs zuzuschreibenden Elementargesetze*, Leipzig, 1873.
Helmholz 'Crelle's Journal,' vol. 72.
Clausius, 'Phil. Mag.,' series 5, Vol. I, p. 69. Vol. X, p. 255.

where A is a constant employed by Helmholtz to denote the ratio of the electrostatic to the electromagnetic unit of electricity, i and i' being expressed in electrostatic units; and κ is an undetermined constant. Also i and i' include as well polarisation currents as true conduction currents.

499.] The first term involving $\frac{\cos\epsilon}{r}$ is the same as the expression for the energy determined by the method of Chap. XVIII, and leads, as we have seen, to $\frac{1}{2}A^2\iiint(Fu+Gv+Hw)\,dx\,dy\,dz$ as the whole electrokinetic energy of the field.

The second term involving $\frac{d^2 r}{ds\,ds'}$ can be shown, on Helmholtz' principles, to depend on the existence of free electricity in the field, and on variation of its potential with the time, and therefore on the existence of unclosed currents, if free electricity in motion have the properties of a current.

We may write

$$i'\frac{dr}{ds'}ds' = \left(u'\frac{dr}{dx'}+v'\frac{dr}{dy'}+w'\frac{dr}{dz'}\right)dx'dy'dz'.$$

And if r denote the distance of the element of volume $dx'dy'dz'$, in which the component currents are u', v', w' from the point x, y, z at which the component currents are u, v, w, then, in forming the expression for the energy of the whole field, the second term becomes

$$\frac{1-\kappa}{4}\iiint\cdot\iiint\cdot\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)\left(u'\frac{dr}{dx'}+v'\frac{dr}{dy'}+w'\frac{dr}{dz'}\right)$$
$$dx\,dy\,dz\,dx'dy'dz'.$$

We shall now assume that a surface S can be described enclosing the field, so distant that the flow of electricity through it, or $(lu+mv+nw)$, where l, m, n are direction cosines of the normal to S, is zero at every point.

Then taking our stand at the point x, y, z, from which the distance is denoted by r, let us form the integral

$$\iiint\left(u'\frac{dr}{dx'}+v'\frac{dr}{dy'}+w'\frac{dr}{dz'}\right)dx'dy'dz'$$

throughout the space within S.

Integrating by parts, it becomes

$$\iint r\,(lu'+mv'+nw')\,dS$$

$$-\iiint r\left(\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}\right)dx'dy'dz', \quad . \quad . \quad . \quad \text{(B)}$$

of which the first term is zero by the condition concerning S.

500.] If all the currents are closed,

$$\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=0$$

at all points, and therefore the second term also vanishes, and therefore also the integral (B) vanishes, and the second term in the expression for the energy of the field vanishes.

But if there be unclosed currents, and if an electric current is equivalent to a transfer of electricity, then in Helmholtz' theory

$$\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}=-\frac{d\rho}{dt},$$

where ρ is the volume density of free electricity in the element $dx'dy'dz'$, and the integral becomes

$$\iiint r\frac{d\rho}{dt}\,dx'dy'dz',$$

that is, $$-\frac{1}{4\pi}\iiint r\nabla^2\frac{dV}{dt}\,dx'dy'dz',$$

if V be the potential of free electricity.

Again, applying Green's theorem to the surface S and the enclosed space, with the functions r and $\frac{dV}{dt}$, we have in the notation of Chap. I,

$$\iint r\frac{d}{d\nu}\frac{dV}{dt}\,dS-\iiint r\nabla^2\frac{dV}{dt}\,dx'dy'dz'$$

$$=\iint\frac{dV}{dt}\frac{dr}{d\nu}\,dS-\iiint\frac{dV}{dt}\nabla^2 r\,dx'dy'dz',$$

of which the two surface integrals vanish if S be distant enough, because whatever free electricity exist in the system, its algebraic sum being zero, $\iint\frac{dV}{dt}\,dS$ must be zero if taken over any sufficiently distant sphere described about x, y, z as centre.

We have then

$$-\frac{1}{4\pi}\iiint r\nabla^2\frac{dV}{dt}\,dx'dy'dz' = -\frac{1}{4\pi}\iiint\frac{dV}{dt}\nabla^2 r\,dx'dy'dz'$$

$$= -\frac{1}{2\pi}\iiint\frac{1}{r}\frac{dV}{dt}\,dx'dy'dz',$$

since $\nabla^2 r = \frac{2}{r}$,

or

$$\iiint\left(u'\frac{dr}{dx'}+v'\frac{dr}{dy'}+w'\frac{dr}{dz'}\right)dx'dy'dz$$

$$= -\frac{1}{2\pi}\iiint\frac{1}{r}\frac{dV}{dt}\,dx'dy'dz',$$

showing the dependence of the second term in the expression for the energy of the field on free electricity and on variation of its potential.

501.] Let us assume

$$-\frac{1}{2\pi}\iiint\frac{1}{r}\frac{dV}{dt}\,dx\,dy\,dz = \Psi,$$

or $\nabla^2\Psi = 2\frac{dV}{dt}$.

Then we proceed as follows. We have

$$\iiint\cdot\iiint\left\{u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right\}\left\{u'\frac{dr}{dx'}+v'\frac{dr}{dy'}+w'\frac{dr}{dz'}\right\}dx\,dy\,dz\,dx'dy'dz'$$

$$= \cdot\iiint\left\{u\frac{d\Psi}{dx}+v\frac{d\Psi}{dy}+w\frac{d\Psi}{dz}\right\}dx\,dy\,dz.$$

Integrating the last expression by parts throughout the space within S, and neglecting the surface integral, it becomes

$$-\iiint\Psi\left\{\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right\}dx\,dy\,dz,$$

that is,

$$\iiint\Psi\frac{d\rho}{dt}\,dx\,dy\,dz,$$

ρ being the volume density of free electricity, that is,

$$-\frac{1}{4\pi}\iiint\Psi\frac{d}{dt}\nabla^2 V\,dx\,dy\,dz,$$

that is

$$-\frac{1}{4\pi}\iiint\Psi\nabla^2\frac{dV}{dt}\,dx\,dy\,dz.$$

Again, by applying Green's theorem to the surface S and the closed space, we have

$$\iint \Psi \frac{d}{d\nu}\frac{dV}{dt}dS - \iiint \Psi \nabla^2 \frac{dV}{dt} dx dy dz$$
$$= \iint \frac{d\Psi}{dt}\frac{dV}{d\nu}dS - \iiint \frac{dV}{dt}\nabla^2 \Psi dx dy dz.$$

And again, neglecting the surface integrals, we have

$$-\frac{1}{4\pi}\iiint \Psi \nabla^2 \frac{dV}{dt} dx dy dz = -\frac{1}{4\pi}\iiint \frac{dV}{dt}\nabla^2 \Psi dx dy dz$$
$$= -\frac{1}{2\pi}\iiint \left(\frac{dV}{dt}\right)^2 dx dy dz.$$

d therefore we have, finally,

$$\iiint \left\{u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right\}\iiint \left\{u'\frac{dr}{dx'} + v'\frac{dr}{dy'} + w'\frac{dr}{dz'}\right\} dx dy dz\, dx' dy' dz'$$
$$-\frac{1}{2\pi}\iiint \left(\frac{dV}{dt}\right)^2 dx dy dz.$$

d the expression for the energy of the field becomes

$$T = A^2 \iiint (Fu + Gv + Hw)\, dx dy dz$$
$$-\frac{A^2}{4\pi}\iiint \left(\frac{dV}{dt}\right)^2 dx dy dz + \frac{A^2 \kappa}{4\pi}\iiint \left(\frac{dV}{dt}\right)^2 dx dy dz.$$

Again,

$$i'\frac{d^2 r}{ds\,ds'} ds\,ds' = \left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\left(u'\frac{dr}{dx'} + v'\frac{dr}{dy'} + w'\frac{dr}{dz'}\right)$$
$$= u\frac{d\Psi}{dx} + v\frac{d\Psi}{dy} + w\frac{d\Psi}{dz}.$$

e expression for the energy of the field becomes then

$$T = A^2 \cdot \iiint \left\{\left(F + \frac{1-\kappa}{2}\frac{d\Psi}{dx}\right)u + \left(G + \frac{1-\kappa}{2}\frac{d\Psi}{dy}\right)v\right.$$
$$\left. + \left(H + \frac{1-\kappa}{2}\frac{d\Psi}{dz}\right)w\right\} dx dy dz.$$

And for the components of electromotive force, so far as they end on the movement of electricity, we have, by Lagrange's ation,

$$-A^2\frac{dF}{dt}-A^2\frac{1-\kappa}{2}\frac{d}{dt}\frac{d\Psi}{dx},$$
$$-A^2\frac{dG}{dt}-A^2\frac{1-\kappa}{2}\frac{d}{dt}\frac{d\psi}{dy},$$
$$-A^2\frac{dH}{dt}-A^2\frac{1-\kappa}{2}\frac{d}{dt}\frac{d\Psi}{dz};$$

to which may be added the statical forces

$$-\frac{dV}{dx}-\frac{dV}{dy}-\frac{dV}{dz}.$$

502.] According to Ohm's law, we have for the diminution of the current by resistance in every element of volume,

$$-Ru=\frac{dV}{dx}+A^2\frac{dF}{dt}+A^2\frac{1-\kappa}{2}\frac{d}{dt}\frac{d\Psi}{dx};$$

and for the heat generated in the element per unit of time,

$$R\,(u^2+v^2+w^2),$$

where R is the resistance at the point where the component currents are u, v, w.

If we write

$$F'=F+\frac{1-\kappa}{2}\frac{d\Psi}{dx},\quad G'=G+\frac{1-\kappa}{2}\frac{d\Psi}{dy},\quad H'=H+\frac{1-\kappa}{2}\frac{d\Psi}{dz},$$

we may obtain the equations of motion in the form given by Helmholtz, viz. in any body presenting electrical resistance,

$$-Ru=\frac{dV}{dx}+A^2\frac{dF'}{dt};$$

and since

$$4\pi u=-\nabla^2F=-\nabla^2F'+\frac{1-\kappa}{2}\nabla^2\frac{d\Psi}{dx}$$
$$=-\nabla^2F'+(1-\kappa)\frac{d^2V}{dx\,dt},$$

$$\left.\begin{aligned}\nabla^2F'-(1-\kappa)\frac{d^2V}{dx\,dt}&=\frac{4\pi}{R}\Big(\frac{dV}{dx}+A^2\frac{dF'}{dt}\Big),\\ \nabla^2G'-(1-\kappa)\frac{d^2V}{dy\,dt}&=\frac{4\pi}{R}\Big(\frac{dV}{dy}+A^2\frac{dG'}{dt}\Big),\\ \nabla^2H'-(1-\kappa)\frac{d^2V}{dz\,dt}&=\frac{4\pi}{R}\Big(\frac{dV}{dz}+A^2\frac{dH'}{dt}\Big).\end{aligned}\right\}$$

Helmholtz' equation in a conductor.

503.] According to this theory if we put

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=\iiint\frac{1}{r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)dx\,dy\,dz$$

$$=-\frac{dV}{dt}, \quad \text{(see Art. 504)}$$

then we have

$$\frac{dF'}{dx}+\frac{dG'}{dy}+\frac{dH'}{dz}=\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}+\frac{1-\kappa}{2}\nabla^2\Psi$$

$$=-\kappa\frac{dV}{dt},$$

since $\nabla^2\Psi=2\dfrac{dV}{dt}$.

According to this theory further the components of magnetic force are

$$a=A\left(\frac{dH'}{dy}-\frac{dG'}{dz}\right)=A\left(\frac{dH}{dy}-\frac{dG}{dz}\right),$$

$$\beta=A\left(\frac{dF}{dz}-\frac{dH}{dx}\right),$$

$$\gamma=A\left(\frac{dG}{dx}-\frac{dF}{dy}\right).$$

And if $\mu=1$,

$$4\pi u=-\nabla^2F'+(1-\kappa)\frac{d^2V}{dx\,dt},$$

$$4\pi v=-\nabla^2G'+(1-\kappa)\frac{d^2V}{dy\,dt},$$

$$4\pi w=-\nabla^2H'+(1-\kappa)\frac{d^2V}{dz\,dt};$$

and

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=A\left(4\pi u+\frac{d^2V}{dx\,dt}\right),$$

$$\frac{da}{dz}-\frac{d\gamma}{dx}=A\left(4\pi v+\frac{d^2V}{dy\,dt}\right),$$

$$\frac{d\beta}{dx}-\frac{da}{dy}=A\left(4\pi w+\frac{d^2V}{dz\,dt}\right).$$

It will be observed that F', G', H' as well as F, G, H are potential functions.

If a moving particle or mass of electricity in motion be equivalent to a current, then according to Helmholtz' formula, unless $\kappa=1$, two such currents or moving masses should have a

potential involving the term $\dfrac{\cos\theta\cos\theta'}{r}$. We are not aware that this is supported by any experimental evidence.

504.] Helmholtz further proves that, if κ be negative, the expression for T may under certain circumstances be negative. To this end we first prove the following theorem :—

For any system

$$\iiint \frac{dF}{dx}\frac{dG}{dy}\,dx\,dy\,dz = \iiint \frac{dF}{dy}\frac{dG}{dx}\,dx\,dy\,dz.$$

For $$\iiint \frac{dF}{dx}\frac{dG}{dy}\,dx\,dy\,dz = \iint F\frac{dG}{dy}\,dS - \iiint F\frac{d^2G}{dx\,dy}\,dx\,dy\,dz,$$

and $$\iiint \frac{dF}{dy}\frac{dG}{dx}\,dx\,dy\,dz = \iint F\frac{dG}{dx}\,dS - \iiint F\frac{d^2G}{dx\,dy}\,dx\,dy\,dz,$$

the surface integrals being taken over the above-mentioned distant surface S. But these surface integrals are of the order -3, and therefore vanish if S be distant enough. Therefore

$$\iiint \frac{dF}{dx}\frac{dG}{dy}\,dx\,dy\,dz = \iiint \frac{dF}{dy}\frac{dG}{dx}\,dx\,dy\,dz.$$

Similarly, $$\iiint \frac{dF}{dx}\frac{dH}{dz}\,dx\,dy\,dz = \iiint \frac{dF}{dz}\frac{dH}{dx}\,dx\,dy\,dz,$$

&c. = &c.

Again, $$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz} = -\frac{dV}{dt}.$$

For since $F = \iiint \dfrac{u'}{r}\,dx'\,dy'\,dz'$, &c., and $\dfrac{d}{dx} = -\dfrac{d}{dx'}$, &c.,

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz} = -\iiint\left\{u'\frac{d}{dx'}+v'\frac{d}{dy'}+w'\frac{d}{dz'}\right\}\frac{1}{r}\,dx'\,dy'\,dz'$$

$$= -\iint \frac{1}{r}u'\,dy'\,dz' - \iint \frac{1}{r}v'\,dx'\,dz' - \iint \frac{1}{r}w'\,dx'\,dy'$$

$$+\iiint \frac{1}{r}\left(\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}\right)dx'\,dy'\,dz',$$

of which the surface integrals are over the surface S, and therefore vanish if that be distant enough.

Therefore

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=\iiint\frac{1}{r}\left(\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}\right)dx'\,dy'\,dz'$$

$$=-\iiint\frac{1}{r}\,\frac{d\rho}{dt}\,dx'\,dy'\,dz',$$

if ρ be the volume density of free electricity

$$=-\frac{dV}{dt}.$$

Again,

$$u=-\frac{1}{4\pi}\nabla^2F,\qquad v=-\frac{1}{4\pi}\nabla^2G,\qquad w=-\frac{1}{4\pi}\nabla^2H,$$

$$\therefore\ \iiint(Fu)\,dx\,dy\,dz=-\frac{1}{4\pi}\iiint F\nabla^2F\,dx\,dy\,dz$$

$$=-\frac{1}{4\pi}\iint F\frac{dF}{dx}\,dy\,dz-\frac{1}{4\pi}\iint F\frac{dF}{dy}\,dx\,dz-\frac{1}{4\pi}\iint F\frac{dF}{dz}\,dx\,dy$$

$$+\frac{1}{4\pi}\iiint\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}dx\,dy\,dz,$$

where $\iint F\frac{dF}{dx}\,dy\,dz+\iint F\frac{dF}{dy}\,dx\,dz+\iint F\frac{dF}{dz}\,dx\,dy$

is taken over the bounding surface S' and vanishes for a sufficiently distant surface.

Therefore

$$\iiint Fu\,dx\,dy\,dz=\frac{1}{4\pi}\iiint\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}dx\,dy\,dz.$$

Similarly

$$\iiint Gv\,dx\,dy\,dz=\frac{1}{4\pi}\iiint\left\{\left(\frac{dG}{dx}\right)^2+\&c.\right\}dx\,dy\,dz,$$

$$\iiint Hw\,dx\,dy\,dz=\frac{1}{4\pi}\iiint\left\{\left(\frac{dH}{dx}\right)^2+\&c.\right\}dx\,dy\,dz.$$

We have then

$$\iiint(Fu+Gv+Hw)\,dx\,dy\,dz$$

$$=\frac{1}{4\pi}\iiint\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}dx\,dy\,dz \quad . \ . \quad (1)$$

$$+\frac{1}{4\pi}\iiint\left\{\left(\frac{dG}{dx}\right)^2+\&c.\right\}dx\,dy\,dz.$$

Also, as we have seen,

$$\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz = \iiint\cdot\left(\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right)^2 dx\,dy\,dz$$

$$=\iiint\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dG}{dy}\right)^2+\left(\frac{dH}{dz}\right)^2\right\} dx\,dy\,dz$$

$$+2\iiint\left(\frac{dF}{dx}\frac{dG}{dy}+\frac{dF}{dx}\frac{dH}{dz}+\frac{dG}{dy}\frac{dH}{dz}\right)dx\,dy\,dz$$

$$=\iiint\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dG}{dy}\right)^2+\left(\frac{dH}{dz}\right)^2\right\} dx\,dy\,dz$$

$$+2\iiint\left(\frac{dF}{dy}\frac{dG}{dx}+\frac{dF}{dz}\frac{dH}{dx}+\frac{dG}{dz}\frac{dH}{dy}\right)dx\,dy\,dz \quad . \quad . \quad . \quad (2)$$

by the theorem above proved.

Multiplying (2) by $\frac{1}{4\pi}$, and then subtracting from (1), we obtain

$$\iiint(Fu+Gv+Hw)\,dx\,dy\,dz$$

$$=\frac{1}{4\pi}\iiint\left\{\left(\frac{dF}{dy}-\frac{dG}{dx}\right)^2+\left(\frac{dG}{dz}-\frac{dH}{dy}\right)^2+\left(\frac{dH}{dx}-\frac{dF}{dz}\right)^2\right\}dx\,dy\,dz$$

$$+\frac{1}{4\pi}\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz.$$

And therefore

$$2T = A^2\iiint(Fu+Gv+Hw)\,dx\,dy\,dz$$

$$-\frac{A^2}{4\pi}\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz+\frac{A^2\kappa}{4\pi}\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz$$

$$=\frac{A^2}{4\pi}\iiint\left\{\left(\frac{dF}{dy}-\frac{dG}{dx}\right)^2+\left(\frac{dG}{dz}-\frac{dH}{dy}\right)^2+\left(\frac{dH}{dx}-\frac{dF}{dz}\right)^2\right\}dx\,dy\,dz$$

$$+\frac{A^2\kappa}{4\pi}\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz.$$

505.] To recapitulate the results of this investigation.— We have

(1) $$\iiint(Fu+Gv+Hw)\,dx\,dy\,dz,$$

$$=\iiint\left\{\left(\frac{dG}{dx}-\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}-\frac{dH}{dx}\right)^2+\left(\frac{dH}{dy}-\frac{dG}{dz}\right)^2\right\}dx\,dy\,dz$$

$$+\frac{1}{4\pi}\iint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz.$$

According to Maxwell's theory this assumption is untrue if u, v, w include the displacement currents.

(2) On Helmholtz' assumption concerning the energy of elementary currents,

$$2T = A^2\iiint(Fu+Gv+Hw)\,dx\,dy\,dz + \frac{(\kappa-1)A^2}{4\pi}\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz$$
$$= A^2\iiint\left\{\left(\frac{dG}{dx}-\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}-\frac{dH}{dx}\right)^2+\left(\frac{dH}{dy}-\frac{dG}{dz}\right)^2\right\}dx\,dy\,dz$$
$$+\frac{A^2\kappa}{4\pi}\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz.$$

Under certain circumstances this expression for T may become negative if κ be negative, as in Weber's theory it is. The controversy whether this is or is not physically possible is discussed in Maxwell's Chapter XXIII. It is possible perhaps to imagine a system of moving electrified masses which shall make T in the above expression negative if κ be negative. In any case where the electrostatic distributions are due to induction, as in the cases treated in Chapters XXI and XXII, the term

$$\frac{A^2\kappa}{4\pi}\iiint\left(\frac{dV}{dt}\right)^2 dx\,dy\,dz$$

will be inappreciable compared with the magnetic energy of conducting circuits, and therefore cannot affect the sign of the total energy.

The difference between the above treatment and Maxwell's consists in the assumption made by Helmholtz and Lorenz, that

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = -\frac{d\rho}{dt}$$

where u, v, w are the components of the total current. According to Maxwell,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$$

is always zero with this meaning of u, v, w.

But $$\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz} = -\frac{d}{dt}\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right) = -\frac{d\rho}{dt},$$

p, q, r being components of the conduction current only.

CHAPTER XXV.

THE ELECTROMAGNETIC THEORY OF LIGHT.

506.] IN Maxwell's theory the electric current at any point in given direction consists of two parts, viz., the current of conduction p, and the time variation of electric displacement $\frac{df}{dt}$.

In the investigation of the induction of electric currents in conductors we have treated of cases in which the displacement currents have no appreciable influence on the conduction currents. We now come to treat of cases in which these conditions are reversed, the conduction currents may be non-existent, the displacement currents having the field to themselves. Maxwell shows that in a medium absolutely non-conducting, but capable of dielectric polarisation, electric disturbances may exist, and are propagated through the medium with velocity varying as $\frac{1}{\sqrt{K\mu}}$, where K is the specific inductive capacity of the medium, and μ its magnetic permeability, and that light consists in the oscillations of dielectric displacement in such a medium with the corresponding magnetic oscillations.

The subject has been treated by H. A. Lorenz, Professor J. J. Thomson, and others.* We proceed to show, following in the main the method elaborated by Lorenz, how some of the phenomena of light may be explained on this hypothesis.

* H. A. Lorenz, *Ueber die Theorie der Reflexion und Refraction des Lichtes Schlömilch Zeitschrift* XXII, XXIII.

J. J. Thomson on Maxwell's *Theory of Light*, Phil. Mag. series 5, vol. ix. p. 284.

Rowland, Phil. Mag., April 1881, June 1884.

Glazebrook, *Report on Optical Theories*, British Association 1885, and works there cited.

Hertz Wiedemann's *Annalen*, 1887–1889.

nuch as the treatment of a simple case is here of excep- importance we shall assume that $\mu = 1$, that is, that no isable matter exists in the medium, so that the magnetic on is identical with the magnetic force. We will also firstly, that there is no conduction. The components of current at any point are then $\dot{f}$, $\dot{g}$, $\dot{h}$, f, g, and h being nponents of dielectric displacement. We then treat them ng all the magnetic properties of an electric current, so

$$\left.\begin{aligned} \dot{f} &= \frac{1}{4\pi}\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right), \\ \dot{g} &= \frac{1}{4\pi}\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right), \\ \dot{h} &= \frac{1}{4\pi}\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right), \end{aligned}\right\} \quad \ldots\ldots \quad \text{(A)}$$

ι, β, γ are the components of magnetic force.
the equation of continuity becomes

$$\left.\begin{aligned} &\frac{d\dot{f}}{dx} + \frac{d\dot{g}}{dy} + \frac{d\dot{h}}{dz} = 0, \\ \text{ence also} \quad &\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0. \end{aligned}\right\} \quad \ldots\ldots \quad \text{(B)}$$

understood that f, g, h, K, and all the other functions ll have to make use of, are expressed in the electro- ic system of units.

Light in an Isotropic Medium.

In an isotropic medium at rest relatively to the source of e shall have at every point

$$\left.\begin{aligned} f &= -\frac{K}{4\pi}\left(\frac{dF}{dt} + \frac{d\psi}{dx}\right) \\ g &= -\frac{K}{4\pi}\left(\frac{dG}{dt} + \frac{d\psi}{dy}\right) \\ h &= -\frac{K}{4\pi}\left(\frac{dH}{dt} + \frac{d\psi}{dz}\right) \end{aligned}\right\}, \quad \ldots\ldots \quad \text{(C)}$$

ψ is the potential of free electricity.

From these we derive the general equations

$$\left.\begin{aligned} K\left(\frac{d}{dt}\frac{d\psi}{dx}+\frac{d^2F}{dt^2}\right)&=\frac{d^2F}{dy^2}+\frac{d^2F}{dz^2}-\frac{d^2G}{dx\,dy}-\frac{d^2H}{dx\,dz},\\ K\left(\frac{d}{dt}\frac{d\psi}{dy}+\frac{d^2G}{dt^2}\right)&=\frac{d^2G}{dz^2}+\frac{d^2G}{dx^2}-\frac{d^2H}{dy\,dz}-\frac{d^2F}{dy\,dx},\\ K\left(\frac{d}{dt}\frac{d\psi}{dz}+\frac{d^2H}{dt^2}\right)&=\frac{d^2H}{dx^2}+\frac{d^2H}{dy^2}-\frac{d^2F}{dx\,dz}-\frac{d^2G}{dy\,dz}.\end{aligned}\right\}\quad\ldots\text{(D)}$$

It may be observed that Helmholtz's system, Art. 498, etc., leads in the case now under consideration to equations of the same form with F', G', H' of Art. 501 for F, G, H.

508.] As we only require a particular solution, we may assume $\psi = 0$,

$$\left.\begin{aligned} f&=p\rho\cos E\\ g&=q\rho\cos E\\ h&=r\rho\cos E\end{aligned}\right\},\quad\ldots\ldots\quad\text{(I)}$$

where ρ is the amplitude of displacement, p, q, r its direction-cosines, and

$$E=\frac{2\pi}{\lambda}\big(vt-(lx+my+nz)\big)=\frac{2\pi}{T}\left(t-\frac{lx+my+mz}{v}\right),$$

T being the periodic time, λ the wave length, and v the wave velocity, so that

$$v\sqrt{K}=1.$$

Also we have

$$\left.\begin{aligned} F&=-\frac{2\lambda}{Kv}p\rho\sin E\\ G&=-\frac{2\lambda}{Kv}q\rho\sin E\\ H&=-\frac{2\lambda}{Kv}r\rho\sin E\end{aligned}\right\}.\quad\ldots\ldots\quad\text{(II)}$$

All points in any plane whose normal is l, m, n are in the same phase at the same time. Any such plane is called the *plane of the wave.*

Since

$$\frac{d\dot{f}}{dx}+\frac{d\dot{g}}{dy}+\frac{d\dot{h}}{dz}=0,\ \text{and}\ \frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,$$

we have

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0;$$

and by equations (I)

$$pl+qm+rn = 0,$$

or the dielectric displacement is in the plane of the wave.

And since $f = p\rho\cos E$, &c., we have

$$\left.\begin{aligned} a &= \frac{4\pi}{K}(rm-qn)\frac{1}{v}\rho\cos E, \\ \beta &= \frac{4\pi}{K}(pn-rl)\frac{1}{v}\rho\cos E, \\ \gamma &= \frac{4\pi}{K}(ql-pm)\frac{1}{v}\rho\cos E. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(III)}$$

And therefore also

$$la+m\beta+n\gamma = 0 \quad \text{and} \quad pa+q\beta+r\gamma = 0,$$

that is to say the magnetic force is in the plane of the wave, and also at right angles to the electromotive force, because the electromotive force in an isotropic medium coincides with the dielectric displacement.

The electrostatic energy per unit of volume is

$$\frac{2\pi}{K}\rho^2\cos^2 E.$$

The electrokinetic energy per unit of volume is

$$\frac{1}{8\pi}(a^2+\beta^2+\gamma^2),$$

which, since $pl+qm+rn = 0$, is also equal to

$$\frac{2\pi}{K}\rho^2\cos^2 E.$$

Hence the energy is half electrostatic and half electrokinetic.

509.] It is thus shewn how a system of waves of electromagnetic disturbance may be propagated through a homogeneous isotropic medium, with velocity $\frac{1}{\sqrt{K}}$. Now, comparing different media, K, or more accurately $K\mu$, in electromagnetic measure, is found to be inversely proportional to the square of the velocity of light, or $\frac{1}{\sqrt{K}}$ with $\mu = 1$ represents the velocity of light in a medium whose specific inductive capacity is K.

Hence it is inferred that light consists of an electromagnetic disturbance.

510.] It comes next in order to explain on this hypothesis the phenomena of reflexion and refraction, when light passes from one isotropic medium into another, separated by a plane from the former.

Let ρ_0, ρ_1, ρ' denote the amplitude of the vibration for the incident, the reflected, and refracted waves respectively; and in like manner any other function shall be distinguished by the suffix or accent according to the wave to which it belongs.

Let us take the plane of incidence for the plane of xz, and the plane of separation for that of yz. In that case,

$$l = \cos\theta, \quad m = 0, \quad n = \sin\theta.$$

Then
$$E = \frac{2\pi}{T}\left(t - \frac{x}{v}\cos\theta - \frac{z}{v}\sin\theta\right),$$

or if the origin be not arbitrary, we must add an arbitrary constant t_0, and write

$$E = \frac{2\pi}{T}\left(t - \frac{x}{v}\cos\theta - \frac{z}{v}\sin\theta - t_0\right),$$

and for the refracted wave

$$E' = \frac{2\pi}{T}\left(t - \frac{x}{v'}\cos\theta' - \frac{z}{v'}\sin\theta' - t_0\right).$$

In order that on the plane of separation $x = 0$ the phase of the refracted light may not differ from that of the incident light, we must have $\dfrac{\sin\theta}{v} = \dfrac{\sin\theta'}{v'}$

or
$$\frac{\sin\theta}{\sin\theta'} = \frac{v}{v'} = \sqrt{\frac{K'}{K}},$$

the well-known law of refraction. This is in no way dependent on the theory as to the nature of light vibrations.

The problem then is, given the direction, θ, of the incident wave, p_0, q_0, r_0 the direction of its dielectric displacement, and ρ_0 the amplitude, to determine the two unknown quantities ρ_1 and ρ', the amplitudes of vibration in the reflected and refracted waves respectively.

For this we require two relations. One we can obtain inde-

pendently of the direction of displacement in the incident wave, that is, of p_0, q_0, r_0; the other depends upon their values.

The energy per unit of volume is proportional to the square of the amplitude of displacement and to $\frac{1}{K}$. The energy that crosses unit area of the plane of separation in unit time towards the refracting medium is, since $\frac{1}{K} = v^2$,

for the incident wave	$\rho_0^2 v^3 \cos\theta$,
for the reflected wave	$-\rho_1^2 v^3 \cos\theta$,
for the refracted wave	$\rho'^2 v'^3 \cos\theta'$.

We may assume that the energy flowing towards the plane is on the whole zero. That is, there being equality of phase for the three waves on the plane of separation,

$$(\rho_0^2 - \rho_1^2) v^3 \cos\theta = \rho'^2 v'^3 \cos\theta',$$

or, having regard to the law of refraction,

$$(\rho_0^2 - \rho_1^2) \sin^3\theta \cos\theta = \rho'^2 \sin^3\theta' \cos\theta'. \quad . \quad . \quad . \quad . \quad (1)$$

This is one relation.

This also expresses the fact that the energy of a wave of the incident light is equal to the sum of the energies of the corresponding waves of the reflected and refracted light.

511.] The form of the second relation will depend on the direction of dielectric displacement in the incident light. We will treat separately the two cases, Case I in which the dielectric displacement is perpendicular to the plane of incidence, or

$$f_0 = 0, \quad h_0 = 0, \quad g_0 = \rho_0 \cos E,$$

and by symmetry $f_1 = 0$, $f' = 0$, &c.; and Case II in which the dielectric displacement is in the plane of incidence; that is,

$$f_0 = -\rho_0 \sin\theta \cos E, \quad g_0 = 0, \quad h_0 = \rho_0 \cos\theta \cos E.$$

Any actual case may then be dealt with by combining the two solutions.

Case I. The electrical theory requires that the electromotive force in either direction parallel to the plane of separation, shall be the same on either side of that plane. Now the electromotive force at right angles to the plane of incidence on the side of

the incident and reflected waves, is $\frac{4\pi}{K}(g_0+g_1)$, and on the side of the refracted wave $\frac{g'}{K'}$, whence we obtain

$$\frac{g_0+g_1}{K}=\frac{g'}{K'},$$

or
$$(g_0+g_1)\,v^2=g'v'^2.$$

That is, neglecting common factors,

$$(\rho_0+\rho_1)\sin^2\theta=\rho'\sin^2\theta'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Combining this with (1), we obtain

$$(\rho_0-\rho_1)\sin\theta\cos\theta=\rho'\sin\theta'\cos\theta',. \quad . \quad . \quad . \quad . \quad (3)$$

and from (2) and (3)

$$\rho_1=\rho_0\frac{\sin(\theta'-\theta)}{\sin(\theta'+\theta)},$$

$$\rho'=\frac{\sin^2\theta}{\sin^2\theta'}(\rho_0+\rho_1).$$

These results agree with those usually given for light optically polarised in the plane of incidence.

COROLLARY

$$a_0+a_1=(\rho_0+\rho_1)\frac{4\pi}{Kv}\sin\theta\cos E,$$

$$a'=\rho'\cdot\frac{4\pi}{K'v'}\sin\theta'\cos E.$$

And therefore

$$a_0+a_1=a'.$$

Similarly it can be shown that

$$\gamma_0+\gamma_1=\gamma',$$

and
$$\beta_0+\beta_1=0=\beta'.$$

That is, the magnetic force does not change discontinuously at the plane of separation.

This result, which we have deduced from (1) and (2), Lorenz treats as an independent relation, and uses it instead of (1).

It will be observed that the energy passing in unit time at any point through unit area of any plane is proportional to the product of the magnetic and electromotive forces at the point and to the sine of the angle between them. See Art. 401, note.

512.] Next, let the dielectric displacement be in the plane of incidence. In that case

$$g_0 = 0, \quad f_0 = -\rho_0 \sin\theta \cos E, \quad h_0 = \rho_0 \cos\theta \cos E,$$

with corresponding values for g_1, &c. Then we have, as before,

$$(\rho_0^2 - \rho_1^2) \sin^3\theta \cos\theta = \rho'^2 \sin^3\theta' \cos\theta'. \quad . \quad . \quad . \quad . \quad (1)$$

For the second condition we take

$$f_0 + f_1 = f',$$

that is, by the electrical theory the dielectric displacement perpendicular to the plane of separation is the same on either side of that plane. This gives

$$(\rho_0 + \rho_1) \sin\theta = \rho' \sin\theta'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Combining (1) and (2), we obtain

$$(\rho_0 - \rho_1) \sin^2\theta \cos\theta = \rho' \sin^2\theta' \cos\theta'. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Whence

$$\rho_1 = \rho_0 \frac{\sin\theta\cos\theta - \sin\theta'\cos\theta'}{\sin\theta\cos\theta + \sin\theta'\cos\theta'} = \rho_0 \frac{\sin 2\theta - \sin 2\theta'}{\sin 2\theta + \sin 2\theta'} = \rho_0 \frac{\tan(\theta' - \theta)}{\tan(\theta' + \theta)},$$

$$\rho' = \frac{\sin\theta}{\sin\theta'}(\rho_0 + \rho_1).$$

These results agree with those usually given for light optically polarised perpendicularly to the plane of incidence. From this and the results above obtained it is inferred that dielectric displacement in the plane of incidence corresponds to optical polarisation perpendicular to that plane, and *vice versa*.

As in the former case, we can deduce from (2) and (3) the continuity of α, β, and γ. Or, following Lorenz's method, assuming the continuity of these functions, we may deduce (1) as a consequence.

If $\frac{\pi}{2} - \omega_0$ be the angle made by the direction of displacement with the common section of the plane of the wave and the plane of incidence, and if ω_1, ω' have corresponding values for the other two waves, we find, combining the two cases,

$$\tan\omega_1 = \tan\omega_0 \frac{\cos\overline{\theta + \theta'}}{\cos\overline{\theta - \theta'}},$$

$$\tan\omega' = \tan\omega_0 \frac{1}{\cos\overline{\theta - \theta'}}.$$

513.] The path of every incident and refracted ray is reversible in direction, so long as the relation

$$\frac{\sin\theta}{\sin\theta'} = \sqrt{\frac{K'}{K}}$$

gives real values for $\cos\theta$ and $\cos\theta'$, so that if light is incident at angle θ' on the right-hand side of the plane of separation, it will be refracted at angle θ on the left-hand side. If, in this case, the amplitude of the polarisation in the incident ray be m_0, and that of the reflected and refracted rays m_1 and m' respectively, we shall have

(1) For dielectric displacement perpendicular to the plane of incidence,

$$m_1 = m_0 \frac{\sin(\theta-\theta')}{\sin(\theta'+\theta)},$$

$$m' = \frac{\sin^2\theta'}{\sin^2\theta}(m_0+m_1).$$

(2) For dielectric displacement in the plane of incidence,

$$m_1 = m_0 \frac{\sin 2\theta' - \sin 2\theta}{\sin 2\theta' + \sin 2\theta},$$

$$m' = \frac{\sin\theta'}{\sin\theta}(m_0+m_1).$$

Comparing these values with those obtained for the direct ray,

$$\rho_1 = \rho_0 \frac{\sin(\theta'-\theta)}{\sin(\theta'+\theta)},$$

$$\rho' = \frac{\sin^2\theta}{\sin^2\theta'}(\rho_0+\rho_1);$$

and

$$\rho_1 = \rho_0 \frac{\sin 2\theta - \sin 2\theta}{\sin 2\theta + \sin 2\theta'},$$

$$\rho' = \frac{\sin\theta}{\sin\theta'}(\rho_0+\rho_1);$$

we see that in both cases

$$m_1 = -\frac{\rho_1}{\rho_0} m_0, \qquad \rho' m' = (\rho_0^2 - m_1^2).$$

If $\frac{v'}{v}\sin\theta > 1$, θ' becomes imaginary, though $\sin\theta' = \frac{v'}{v}\sin\theta$ remains real. For these values of θ we have total reflexion. For application of the theory to the resulting phenomena see Lorenz's treatise above referred to.

514.] We have assumed the medium to be at rest. We may however conceive it to be moving relatively to the source of light with velocity V, very small compared with the velocity $\frac{1}{\sqrt{K}}$, in the direction of wave motion. The effect of a motion of the medium in the plane of the wave will not be considered.

Let the direction of wave motion be that of y, the direction of displacement that of x, we shall then have by the field equations Chap XXI,

$$f = \frac{K}{4\pi}\left\{V\gamma - \frac{dF}{dt}\right\}$$

$$\frac{df}{dy} = \frac{K}{4\pi}\left(V\frac{d\gamma}{dy} - \frac{d}{dt}\frac{dF}{dy}\right) = \frac{K}{4\pi}\left(V\frac{d\gamma}{dy} + \frac{d\gamma}{dt}\right).$$

Also in this case $\frac{d\beta}{dz} = 0$, and

$$\frac{d\gamma}{dy} = 4\pi\frac{df}{dt};$$

whence $$\frac{d^2f}{dy^2} = K\left(\frac{d^2f}{dt^2} + V\frac{d}{dy}\frac{df}{dt}\right).$$

Assume $f = a\cos E$

and $$E = \frac{2\pi}{T}\left(t - \frac{y}{v} + t'\right)$$

where t' is a constant.

This gives $$\frac{1}{v^2} = K\left(1 - \frac{V}{v}\right),$$

and $$v = \frac{V}{2} \pm \sqrt{\frac{1}{K} + \frac{V^2}{4}},$$

in which the $+$ sign must be taken, and V being very small compared with $\frac{1}{\sqrt{K}}$, $v = \frac{1}{\sqrt{K}} + \frac{V}{2}$, or the velocity of the light is increased by half the velocity of the medium relative to the source.*

On the Passage of a Wave through a Partially Conducting Medium.

515.] If the conductivity be finite instead of zero as hitherto supposed, Maxwell's theory gives us

* See a paper by Professor J. J. Thomson, 'Phil. Mag.' 1880.

$$p = CP,\ f = \frac{K}{4\pi}P,\ u = p+\dot{f},\ \text{etc.};$$

also $$P = -\frac{dF}{dt} - \frac{d\psi}{dx}, \qquad F = \iiint \frac{u}{r}\,dxdydz,$$

$$\psi = \frac{1}{K}\iiint \frac{1}{r}\left\{\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right\}dxdydz;$$

and therefore $$\nabla^2\psi = -\frac{4\pi}{K}\left\{\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right\}.$$

Also $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$, and therefore $\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$

From these equations we get

$$u = \frac{4\pi C}{K}f + \dot{f},\ \nabla^2 F = -4\pi\left\{\frac{4\pi C}{K}f + \dot{f}\right\};$$

$$\therefore\ \nabla^2 f = \frac{K}{4\pi}\nabla^2 P = \frac{K}{4\pi}\left\{-\frac{d}{dt}\nabla^2 F - \frac{d}{dx}\nabla^2\psi\right\};$$

and therefore $$\nabla^2 f = 4\pi C\frac{df}{dt} + K\frac{d^2 f}{dt^2} + \frac{d}{dx}\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right). \quad . \quad . \quad (1)$$

Differentiating equation (1) with regard to x, and the corresponding equations in g and h with regard to y and z respectively, and adding, we obtain

$$4\pi C\frac{d}{dt}\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) + K\frac{d^2}{dt^2}\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) = 0.$$

For a wave normal to x, all the functions $\frac{df}{dy}$, $\frac{df}{dz}$, $\frac{dg}{dy}$, etc. are zero. And therefore we have for the normal displacement f, in such a wave

$$\nabla^2 f = \frac{d^2 f}{dx^2},$$

and therefore from (1) $$4\pi C\frac{df}{dt} + K\frac{d^2 f}{dt^2} = 0.$$

Therefore either $f = 0$ throughout, or else f varies as $\epsilon^{-\lambda t}$.

Now if the conducting medium be bounded by the plane of yz, light flowing from that plane, and if on that plane the functions be periodic, they must be periodic at all points within the conducting medium. In any such case then f must be zero

throughout, or, given Maxwell's formulæ, there can be no normal displacement.

For the transversal displacements we have

$$\nabla^2 g = 4\pi C \frac{dg}{dt} + K \frac{d^2 g}{dt^2},$$

that is
$$\frac{d^2 g}{dx^2} = 4\pi C \frac{dg}{dt} + K \frac{d^2 g}{dt^2}.$$

We may assume as solution

$$g = q\rho\epsilon^{-2\pi vCx} \cos \frac{2\pi}{T}\left(t - \frac{x}{v}\right),$$

where T is the periodic time; whence

$$v^2 = -\frac{K}{2C^2T^2} \pm \sqrt{\frac{K^2+4C^2T^2}{4C^4T^4}},$$

in which the positive sign must be taken.

And if $\frac{C^2}{K}$ be very small,

$$\frac{1}{v^2} = K + \frac{C^2T^2}{K}.$$

We may call a wave in which at any given point all the functions are periodic functions of the time, but affected by the factor $\epsilon^{-2\pi vCx}$, a *stationary wave*. We see that if the wave be stationary, it can, on Maxwell's theory, have no normal displacement.

516.] Let us now consider the case of a wave of light passing from an isotropic non-conducting medium into a partially conducting medium, separated from the former by the plane of yz. Let the plane of incidence be that of xz, and let us take the case of optical polarisation in the plane of incidence, or the dielectric displacement in direction y.

If the dielectric displacement in the conducting medium be $\rho \cos E$, the magnetic force in that medium consists of two parts, one derived from the displacement current, and the other from the conduction current. The value of its z-component is then (using the same notation as before)

$$\frac{4\pi}{\sqrt{K'}}\left(\rho' \cos E + \frac{2TC}{K'} \rho' \sin E\right) \cos \theta'.$$

517.] We shall now find, as the consequence of our theory, that, if the magnetic and electromotive forces be continuous, as we found them to be between two insulating media, there must be a difference of phase between the reflected, the refracted, and the incident wave at any point on the plane of separation.

For let us assume

for the incident wave, $g = a\cos E$;

for the reflected wave, $g = a_1\cos E + b_1\sin E$;

for the refracted wave, $g = a_2\cos E + b_2\sin E$.

Then we shall have

(1) by the continuity of electromotive force in y,

$$((a+a_1)\cos E + b_1\sin E)\sin^2\theta = (a_2\cos E + b_2\sin E)\sin^2\theta';$$

(2) by the continuity of magnetic force in z,

$$(\overline{a-a_1}\cos E - b_1\sin E)\sin\theta\cos\theta = (a_2\cos E + b_2\sin E)\sin\theta'\cos\theta'$$
$$+ (a_2\sin E - b_2\cos E)\frac{2TC}{K'}\sin\theta'\cos\theta'.$$

Equating coefficients of $\cos E$ and $\sin E$, we have four equations to determine the four unknown quantities $a_1\ b_1$, $a_2\ b_2$.

The existence of the second term in the right-hand member of (2), which is introduced by the conduction, forbids us to make $b_1 = b_2 = 0$, which would reduce the three waves to the same phase on the plane of separation. We have in fact a difference of phase between the incident and reflected wave $\tan^{-1}\frac{b_1}{a_1}$, and between the incident and refracted wave $\tan^{-1}\frac{b_2}{a_2}$.

518.] We have preferred thus far to use real quantities as far as practicable. But the treatment of problems of this class is frequently much facilitated by the employment of the exponential instead of the circular function. As for instance in the case of

Reflexion from a metallic surface.

As the medium treated of in the last article becomes a pure conductor, let us replace the circular function $\cos E$ by the corresponding exponential form. Let us then suppose light passing through a dielectric or perfectly insulating medium and incident on a metallic surface. Let the surface of separation be the plane

of xz, and the plane of incidence that of yz. Let the light be optically polarised in the plane of incidence.

Then for the incident light we have a displacement

$$\xi = \rho_0 \epsilon^{-E\sqrt{-1}}$$

and $$E = \frac{2\pi}{T}\left(t - \frac{y}{V}\cos\theta - \frac{z}{V}\sin\theta - t_0\right).$$

For the reflected light

$$\xi_1 = \rho_1 \epsilon^{-E_1\sqrt{-1}}$$

and $$E_1 = \frac{2\pi}{T}\left(t + \frac{y}{V}\cos\theta - \frac{z}{V}\sin\theta - t_1\right).$$

Within the metal we shall have in lieu of the displacement a conduction current u, where $u = \rho' \epsilon^{-E'\sqrt{-1}}$

and $$E' = \frac{2\pi}{T}\left(t - \frac{y}{V'}\cos\theta' - \frac{z}{V'}\sin\theta' - t'\right);$$

or assuming the law $\frac{\sin\theta'}{V'} = \frac{\sin\theta}{V}$ to hold in this case

$$E' = \frac{2\pi}{T}\left(t - \frac{y}{V'}\cos\theta' - \frac{z}{V}\sin\theta - t'\right),$$

in which V' and θ', and therefore ρ' and ρ_1, are wholly or in part imaginary.

If now the conditions assumed in Arts. 510, 511 remain formally true with these symbolical values of the variables, we shall have by the continuity of the electromotive force in x on the plane of separation $y = 0$,

$$\frac{\rho_0 + \rho_1}{K} = \sigma\rho', \quad \ldots \ldots \quad (1\,a)$$

$4\pi\sigma$ being resistance, and by the continuity of the magnetic force in z

$$(\rho_0 - \rho_1)\sin\theta\cos\theta = \sigma\rho'\cot\theta'; \quad \ldots \ldots \quad (3\,a)$$

whence $$\frac{\rho_0 - \rho_1}{\rho_0 + \rho_1}\cot\theta = \cot\theta',$$

or $$\rho_1 = \rho_0 \frac{\sin(\theta - \theta')}{\sin(\theta + \theta')}, \text{ as in Art. 511.}$$

It is usual to determine the intensity, ρ_1, of the vibration in the reflected light by taking the real part of this expression.

The treatment of the problem in the form now presented is not peculiar to the electromagnetic theory of the nature of light. We therefore follow it no further.

519.] Helmholtz supposes the dielectric to be capable of molecular polarisation, and investigates the laws of propagation of this polarisation, arriving at results analogous to those of Maxwell.

In his view each molecule of the dielectric in a field of electromotive force becomes polarised, or charged with equal and opposite amounts of electricity proportional to the electromotive force, these polarisations being of the same nature as those of small conductors, so that representing the amount per unit area on a plane at any point normal to the resultant force by σ, this σ is of opposite sign to Maxwell's displacement.

If f, g, h be the components of polarisation at any point, the electrical density of polarisation is

$$-\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right).$$

Helmholtz supposes the variations of these polarisations to possess the electromagnetic properties of ordinary currents, as we have supposed with regard to Maxwell's displacement currents. In this theory therefore if the only electricity in the dielectric be that arising from polarisation, and there be no conduction, we have

$$F = A^2\iiint\frac{\dot{f}}{r}\,dx\,dy\,dz+\frac{1-k}{2}\frac{d\phi}{dx},$$

$$f=\epsilon\left(-\frac{dF}{dt}-\frac{d\psi}{dx}\right),$$

where ϵ is the constant ratio of polarisation to electromotive force, and where

$$\nabla^2\psi = 4\pi\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right),\text{ and } \phi = -\frac{1}{2\pi}\iiint\frac{1}{r}\frac{d\psi}{dt}\,dx\,dy\,dz,$$

as given in Chap. XXIV.

In this theory however J, or $\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}$, is not zero, but, as above shown, is equal to $-k\frac{d\psi}{dt}$

Hence we get

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=\epsilon\Big(-\frac{dJ}{dt}-\nabla^2\psi\Big);$$

or representing $\dfrac{df}{dx}+\dfrac{dg}{dy}+\dfrac{dh}{dz}$ by j,

$$\epsilon\frac{dJ}{dt}=-(1+4\pi\epsilon)\,j.$$

Again,

$$\begin{aligned}\nabla^2 f &= -\epsilon\frac{d}{dt}\nabla^2 F-\epsilon\frac{d}{dx}\nabla^2\psi\\ &= 4\pi\epsilon A^2\frac{d^2 f}{dt^2}-(1-k)\,\epsilon\frac{d^3\psi}{dx\,dt^2}-4\pi\epsilon\frac{dj}{dx}\\ &= 4\pi\epsilon A^2\frac{d^2 f}{dt^2}-\frac{1-k}{k}(1+4\pi\epsilon)\frac{dj}{dx}-4\pi\epsilon\frac{dj}{dx}\\ &= 4\pi\epsilon A^2\frac{d^2 f}{dt^2}+\Big\{1-\frac{1+4\pi\epsilon}{k}\Big\}\frac{d}{dx}\Big(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\Big),\end{aligned}$$

whence there would appear to be a wave of normal disturbance with velocity

$$\frac{1}{A}\sqrt{\frac{1+4\pi\epsilon}{4\pi\epsilon k}},$$

and a wave of transversal disturbance with velocity

$$\frac{1}{A\sqrt{4\pi\epsilon}}.$$

If however the medium be absolutely non-conducting we must necessarily have ψ independent of t or $\dfrac{d\dot f}{dx}+\dfrac{d\dot g}{dy}+\dfrac{d\dot h}{dz}=0$; whence it would appear that the normal disturbance is constant. It is difficult therefore to interpret Helmholtz's theory of propagation of electric polarisation through the dielectric unless we assume a passage of electricity independent of conduction, that is to say a convective passage not obeying Ohm's law.

In fact if ψ be independent of the time, J becomes zero, and it is hard to distinguish between Helmholtz's theory and that of Maxwell.

Anisotropic Media.

520.] We now proceed to the case of anisotropic media. In any such medium we have generally three different values of K,

corresponding to three mutually perpendicular directions in space, fixed in the medium. We assume the medium to be homogeneous, and these directions, with the corresponding values of K, the same at all points.

Taking these directions for axes of x, y, and z, we will denote by K_x, K_y, and K_z the corresponding values of K. We shall assume, as before, that $\mu = 1$.

Our equations (C) then become

$$\left.\begin{aligned} \frac{4\pi f}{K_x} &= -\frac{dF}{dt} - \frac{d\psi}{dx} \\ \frac{4\pi g}{K_y} &= -\frac{dG}{dt} - \frac{d\psi}{dy} \\ \frac{4\pi h}{K_z} &= -\frac{dH}{dt} - \frac{d\psi}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (C')$$

As in the case of an isotropic medium, the condition of continuity requires that

$$\left.\begin{aligned} &\frac{d\dot{f}}{dx} + \frac{d\dot{g}}{dy} + \frac{d\dot{h}}{dz} = 0, \\ \text{and therefore} \quad &\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

From the equations (C′) combined with

$$\left.\begin{aligned} 4\pi\dot{f} &= \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \text{ etc.,} \\ \text{and} \quad a &= \frac{dH}{dy} - \frac{dG}{dz}, \text{ etc.,} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

we obtain the general equations

$$\left.\begin{aligned} K_x\left(\frac{d^2F}{dt^2} + \frac{d}{dt}\frac{d\psi}{dx}\right) &= \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2} - \frac{d^2G}{dxdy} - \frac{d^2H}{dxdz}, \\ K_y\left(\frac{d^2G}{dt^2} + \frac{d}{dt}\frac{d\psi}{dy}\right) &= \frac{d^2G}{dz^2} + \frac{d^2G}{dx^2} - \frac{d^2H}{dydz} - \frac{d^2F}{dxdy}, \\ K_z\left(\frac{d^2H}{dt^2} + \frac{d}{dt}\frac{d\psi}{dz}\right) &= \frac{d^2H}{dx^2} + \frac{d^2H}{dy^2} - \frac{d^2F}{dxdz} - \frac{d^2G}{dydz}. \end{aligned}\right\} \quad . \quad (D')$$

We may assume as a solution $\psi = 0$,

$$\left.\begin{aligned} f &= p\rho\cos E, \\ g &= q\rho\cos E, \\ h &= r\rho\cos E; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (Ia)$$

where $E = \frac{2\pi}{\lambda}(vt - (lx+my+nz)) = \frac{2\pi}{T}\left(t - \frac{lx+my+nz}{v}\right)$,

$$\left.\begin{aligned} F &= -\frac{2\lambda}{K_x v} p\rho \sin E, \\ G &= -\frac{2\lambda}{K_y v} q\rho \sin E, \\ H &= -\frac{2\lambda}{K_z v} r\rho \sin E. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II}a)$$

As before, λ denotes the wave length, and v here denotes the velocity of a wave in direction l, m, n. As we shall see, v is generally different for different directions.

521.] The conditions (A) involve, as in the case of an isotropic medium,

$$pl + qm + rn = 0,$$

or the dielectric displacement is in the plane of the wave.

And substituting in (D′) the values assumed above for F, G, H, we have

$$\left.\begin{aligned} v^2 p &= (m^2+n^2)\frac{p}{K_x} - ln\frac{r}{K_z} - lm\frac{q}{K_y} \\ v^2 q &= (l^2+n^2)\frac{q}{K_y} - ml\frac{p}{K_x} - mn\frac{r}{K_z} \\ v^2 r &= (l^2+m^2)\frac{r}{K_z} - nm\frac{q}{K_y} - nl\frac{p}{K_x} \end{aligned}\right\}; \quad . \quad . \quad . \quad (1)$$

that is

$$\left.\begin{aligned} v^2 p &= \frac{p}{K_x} - l\left(\frac{pl}{K_x} + \frac{qm}{K_y} + \frac{rn}{K_z}\right) \\ v^2 q &= \frac{q}{K_y} - m\left(\frac{pl}{K_x} + \frac{qm}{K_y} + \frac{rn}{K_z}\right) \\ v^2 r &= \frac{r}{K_z} - n\left(\frac{pl}{K_x} + \frac{qm}{K_y} + \frac{rn}{K_z}\right) \end{aligned}\right\}; \quad . \quad . \quad . \quad (2)$$

and multiplying these equations in order by p, q, and r, and remembering that

$$pl + qm + rn = 0,$$

we have

$$v^2 = \frac{p^2}{K_x} + \frac{q^2}{K_y} + \frac{r^2}{K_z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{E})$$

And therefore v, the velocity of wave motion in direction l, m, n, is inversely proportional to the radius vector ρ in direction p, q, r to the ellipsoid

$$\frac{x^2}{K_x} + \frac{y^2}{K_y} + \frac{z^2}{K_z} = 1.$$

The ellipsoid is then determinate in the direction and magnitude of its axes, and the same at all points in the homogeneous medium. We shall call this ellipsoid the *dielectric ellipsoid.*

522.] If we seek from equations (2) to determine the ratios $p:q:r$, we have to eliminate v^2 and $\left(\frac{pl}{K_x}+\frac{qm}{K_y}+\frac{rn}{K_z}\right)$.

As the result we obtain the determinantal equation

$$\begin{vmatrix} \frac{p}{K_x}, & \frac{q}{K_y}, & \frac{r}{K_z} \\ p, & q, & r \\ l, & m, & n \end{vmatrix} = 0;$$

from which, combined with

$$pl+qm+rn=0,$$

the ratios $\frac{p}{r}$ and $\frac{q}{r}$ can be determined.

Again, the section made by the plane

$$lx+my+nz=0,$$

with the dielectric ellipsoid is an ellipse, which we will call the ellipse LMN: and if we seek to determine the direction of its axes, we make $x^2+y^2+z^2$ maximum or minimum consistently with

$$\frac{x^2}{K_x}+\frac{y^2}{K_y}+\frac{z^2}{K_z}=1,$$

and $$lx+my+nz=0.$$

This gives the same determinant as before. Hence we see that for given direction l, m, n of the wave motion, the direction of dielectric displacement p, q, r must be one or other axis of the ellipse LMN, and v, the wave velocity, is inversely proportional to that axis. A wave may move in the given direction l, m, n with either of two velocities according as the direction of displacement is that of one or the other axis of the ellipse LMN.

If, however, all points in the plane perpendicular to l, m, n be given in the same phase of displacement at the same instant, and that displacement in any other direction in the plane than either of the two axes of the ellipse LMN, the displacement cannot be propagated as a single wave. We must resolve it

into two components parallel to the two axes of the ellipse LMN. Each component of displacement is then propagated as a separate wave with velocity inversely proportional to the axis of the ellipse LMN to which it is parallel.

Of the Magnetic Force.

523.] The components of the magnetic force are

$$\left.\begin{aligned} \alpha &= \frac{dH}{dy} - \frac{dG}{dz} = \frac{4\pi}{v}\rho\cos E\left(m\frac{r}{K_z} - \frac{nq}{K_y}\right), \\ \beta &= \frac{4\pi}{v}\rho\cos E\left(n\frac{p}{K_x} - l\frac{r}{K_z}\right), \\ \gamma &= \frac{4\pi}{v}\rho\cos E\left(l\frac{q}{K_y} - m\frac{p}{K_x}\right). \end{aligned}\right\} \quad (\text{III}\,a)$$

And therefore $l\alpha + m\beta + n\gamma = 0,$

or the magnetic force is perpendicular to the wave direction, and, as will be seen in the next Article, it is also perpendicular to the displacement and the electromotive force.

Of the Electromotive Force.

524.] The direction-cosines of the electromotive force are proportional to $\frac{p}{K_x}$, $\frac{q}{K_y}$, $\frac{r}{K_z}$.

So also are the direction-cosines of the perpendicular from O on the tangent plane to the dielectric ellipsoid through the extremity of the radius vector p, q, r. If therefore the dielectric displacement be in the radius vector OP of the dielectric ellipsoid, the electromotive force is in the perpendicular from O on the tangent plane at P.

The vanishing of the determinant Art. 522, shows that the three vectors whose direction-cosines are proportional to $\frac{p}{K_x}$, $\frac{q}{K_y}$ and $\frac{r}{K_z}$, p, q and r, l, m and n; that is, the electromotive force, the dielectric displacement, and the direction of wave motion, are in one plane. The components of displacement are $p\rho$, $q\rho$, $r\rho$. Those of electromotive force are $\rho\frac{p}{K_x}$, $\rho\frac{q}{K_y}$, $\rho\frac{r}{K_z}$.

Hence the component of electromotive force in the direction of the displacement is

$$\rho\left(\frac{p^2}{K_x}+\frac{q^2}{K_y}+\frac{r^2}{K_z}\right),$$

that is ρv^2. And since OP, OQ, and the direction of wave motion are coplanar, that is the resolved part of the electromotive force in the plane of the wave. But the electromotive force has a component normal to the plane of the wave namely $\rho v^2 \tan POQ$, that is $\rho v^2 \tan \epsilon$, suppose.

525.] Now let the plane of the displacement, the electromotive force and the wave motion be that of the figure. Let OP be the dielectric displacement, and therefore an axis of the ellipse LMN which is perpendicular to the figure. Let OQ be the perpendicular on the tangent plane through P to the dielectric ellipsoid. Let OS be the normal l, m, n to the ellipse LMN. If we make $OS = OP$, we find from (E) that the locus of S is

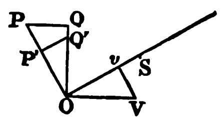

Fig. 49.

$$\frac{K_x x^2}{\rho^2-K_x}+\frac{K_y y^2}{\rho^2-K_y}+\frac{K_z z^2}{\rho^2-K_z}=0,$$

that is, $$\frac{x^2}{\rho^2-K_x}+\frac{y^2}{\rho^2-K_y}+\frac{z^2}{\rho^2-K_z}=1,$$

where $$\rho^2 = x^2+y^2+z^2.$$

526.] Let us now invert the system with unit radius of inversion. Let P then become P', and Q, Q'. Then $OP'=v$. And let us take a point v in OS such that $Ov = OP' = v$.

At the same time let

$$\frac{1}{K_x}=A^2,\quad \frac{1}{K_y}=B^2,\quad \frac{1}{K_z}=C^2.$$

Then (1), the equation to the locus of v, is

$$\frac{x^2}{v^2-A^2}+\frac{y^2}{v^2-B^2}+\frac{z^2}{v^2-C^2}=0.$$

We will call this the v-surface. It is such that the radius vector to it at any point is the wave velocity in the direction of that radius vector.

(2) The equation to the locus of Q' is the new ellipsoid

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1.$$

(3) The locus of P' is the locus of the foot of the perpendicular from the centre on the tangent plane to this new ellipsoid.

(4) If a plane perpendicular to that of the figure be drawn through OQ, its intersection with the new ellipsoid is an ellipse of which OQ' is an axis.

(5) Therefore if in the plane of the figure we draw $OV \perp OQ'$, and make $OV = OQ'$, the equation to the locus of V is, as above shown,

$$\frac{x^2}{V^2 - A^2} + \frac{y^2}{V^2 - B^2} + \frac{z^2}{V^2 - C^2} = 1.$$

We will call this the V-surface.

(6) Since a plane through $P'Q'$ perpendicular to the figure is a tangent plane to the new ellipsoid or locus of Q', a plane through Vv perpendicular to the figure is a tangent plane to the V-surface or locus of V. The v-surface is therefore the locus of the foot of the perpendicular from the centre on the tangent plane to the V-surface. The reader may verify analytically that the surface whose equation is

$$\frac{x^2}{v^2 - A^2} + \frac{y^2}{v^2 - B^2} + \frac{z^2}{v^2 - C^2} = 0,$$

with $$v^2 = x^2 + y^2 + z^2,$$

is the locus of the foot of the perpendicular from the centre on the tangent plane to the surface whose equation is

$$\frac{x^2}{V^2 - A^2} + \frac{y^2}{V^2 - B^2} + \frac{z^2}{V^2 - C^2} = 1,$$

with $$V^2 = x^2 + y^2 + z^2.$$

527.] The V-surface above determined is known in physical optics as the *wave surface*, and from the preceding reasoning it follows that the well-known application of its properties and those of the v-surface to the determination of wave and ray velocity, and the magnitude and direction of ethereal vibration holds, mutatis mutandis, in the case of the propagation of displacements in the dielectric.

It follows, for instance, that OV, the radius vector to the V-surface, is the velocity with which a disturbance originating at O is propagated in direction OV. Let all points in a plane through O at right angles to Ov be in the same phase at the same instant, that is, assume the wave front to be perpendicular to Ov. Then the plane Vv is also a wave front, and the line OV is the line of quickest passage of the disturbance from one wave front to another.

528.] We can find by a known construction the direction of the refracted wave when light passes from an isotropic into an anisotropic medium separated by a plane from the former—or rather the path of each of the two refracted waves—since the two surfaces which we have called the v-surface and the V-surface respectively can be formed from either axis of the ellipse LMN, and each has therefore two sheets.

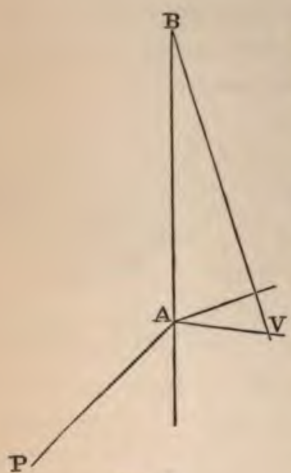

Fig. 50.

Let AB be the plane of separation, PA the direction of the incident light in the isotropic medium.

Let the plane of incidence be that of the figure.

Then the angle of incidence, $PAB - \frac{\pi}{2}$, being known, we know the position of B in AB such that B shall be the first point in direction AB which is in the same phase with A.

About A as centre describe the V-surface. Through B suppose a line drawn perpendicular to the plane of the figure, and through this line a tangent plane to the V-surface touching it in V.

Let Ov be the perpendicular from O on this tangent plane. Then Ov is the direction of wave motion of the refracted ray, OV the direction in which the disturbance is propagated, that is the direction of the ray. Ov is in the plane of the figure, OV not generally so.

As the V-surface has two sheets, there are generally two directions of Ov and OV for given direction PA of the incident light.

We see then that, given the angle of incidence θ, there are two determinate directions of wave motion in the crystalline medium, and therefore two determinate angles of refraction, θ'_1 and θ'_2, one for each of the two refracted waves. And each obeys the law of refraction $\frac{v}{v'} = \frac{\sin\theta}{\sin\theta'}$, thus preserving continuity of phase.

529.] The direction of wave motion for either refracted ray being now determined, the direction of its dielectric displacement is also determined by Art. 522, and is independent of the polarisation of the incident light.

But although the direction of displacement in either refracted ray is independent of the polarisation of the incident light, the amplitude is not. And, as we shall see, it is possible by suitably choosing the plane of polarisation of the incident light to reduce to zero the amplitude for either refracted ray, so that only one of the two refracted rays will exist. Suppose ω to be the angle made by the plane of polarisation of the incident light with the normal to the plane of incidence. Then for a certain value of ω, suppose ω_a, one of the two refracted waves disappears, and for ω_b the other disappears. Then any given displacement in the incident light may be resolved in directions denoted by ω_a and ω_b; and the component in ω_a gives rise exclusively to one, that in ω_b exclusively to the other, refracted ray.

Let ω_1 be the angle made by the dielectric displacement of the reflected light with the normal to the plane of incidence, ω' the same for the single refracted wave, ω being so chosen that there shall be only one. Also let ρ_0, ρ_1, ρ' denote the amplitudes of displacement for the three waves respectively. Then ρ_0, θ, θ_1, θ' and ω' are given. And we have to find ω, ω_1, ρ_1, and ρ'. The value of ω so found is the value ω_a, which causes the other refracted ray to disappear.

530.] To determine these four quantities we require four equations. One is the equation of the flow of energy analogous to that

of Art. 510. The flow of energy towards the plane of separation on the incident side is, as in that article, $(\rho_0^2-\rho_1^2)\, v^3 \cos\theta$.

The energy per unit of volume in the anisotropic medium is $v'^2\rho'^2$, by Art. 521, where ρ' is the amplitude of dielectric displacement. The direction and velocity with which it flows are represented by OV, and the projection of OV on the normal to the plane of separation is

$$v'\cos\theta' + v'\sin\theta'\sin\omega'\tan\epsilon,$$

since $\epsilon = POV = VOv$. Therefore the flow of energy on this side normal to the plane of separation is

$$\rho'^2(v'^3\cos\theta' + v'^3\sin\theta'\sin\omega'\tan\epsilon)\,;$$

and therefore for the case in which only one refracted ray exists our equation becomes

$$(\rho_0^2-\rho_1^2)\sin^3\theta\cos\theta = \rho'^2\{\sin^3\theta'\cos\theta' + \sin^4\theta'\sin\omega'\tan\epsilon\}. \quad . \quad . \quad (1)$$

The electromagnetic theory provides us with three more equations, namely:—

By the continuity of electromotive force perpendicular to the plane of incidence

$$(\rho_0\cos\omega_0 + \rho_1\cos\omega_1)\sin^2\theta = \rho'\cos\omega'\sin^2\theta'. \quad . \quad . \quad . \quad (2)$$

By the continuity of electromotive force in the common section of the planes of incidence and separation,

$$(\rho_0\sin\omega_0 - \rho_1\sin\omega_1)\sin^2\theta\cos\theta = \rho'\sin\omega'\sin^2\theta'\cos\theta' + \rho'\sin^3\theta'\tan.\epsilon \quad (3)$$

By the continuity of dielectric displacement perpendicular to the plane of separation,

$$(\rho_0\sin\omega_0 + \rho_1\sin\omega_1)\sin\theta = \rho'\sin\omega'\sin\theta'. \quad . \quad . \quad . \quad (4)$$

From (3) and (4) we obtain

$$(\rho_0^2\sin^2\omega_0 - \rho_1^2\sin^2\omega_1)\sin^3\theta\cos\theta$$
$$= \rho'^2\sin^2\omega'\sin^3\theta'\cos\theta' + \rho'^2\sin^4\theta'\sin\omega'\tan\epsilon\,;$$

and subtracting this from (1),

$$(\rho_0^2\cos^2\omega_0 - \rho_1^2\cos^2\omega_1)\sin^3\theta\cos\theta = \rho'^2\cos^2\omega'\sin^3\theta'\cos\theta'^2;$$

and dividing by (2),

$$(\rho_0\cos\omega_0 - \rho_1\cos\omega_1)\sin\theta\cos\theta = \rho'\cos\omega'\sin\theta'\cos\theta'. \quad . \quad . \quad (5)$$

And we have now four linear equations (2), (3), (4), and (5) from which to determine ω_0 and ω_1, ρ_1 and ρ'.

531.] Eliminating $\rho_1 \cos\omega_1$ from (2) and (5), we obtain

$$2\rho_0 \cos\omega_0 \sin^2\theta\cos\theta = \rho'\cos\omega'(\sin^2\theta'\cos\theta + \sin\theta\sin\theta'\cos\theta'),$$
$$= \rho'\cos\omega'\sin\theta'\sin(\theta'+\theta).$$

Eliminating $\rho_1 \sin\omega_1$ from (3) and (4), we obtain

$$2\rho_0 \sin\omega_0\sin^2\theta\cos\theta$$
$$= \rho'\sin\omega'(\sin^2\theta'\cos\theta' + \sin\theta'\sin\theta\cos\theta) + \rho'\sin^3\theta'\tan\epsilon;$$

and therefore

$$\tan\omega_0 = \tan\omega'\frac{\sin\theta'\cos\theta' + \sin\theta\cos\theta}{\sin(\theta+\theta')} + \frac{\sin^2\theta'\tan\epsilon}{\cos\omega'\sin\overline{\theta+\theta'}}$$
$$= \tan\omega'\cos\overline{\theta-\theta'} + \frac{\sin^2\theta'\tan\epsilon}{\cos\omega'\sin\overline{\theta+\theta'}}.$$

This reduces to the formula of Art. 512 if the medium be isotropic, because in this case $\epsilon = 0$.

Again, eliminating $\rho_0\cos\omega_0$ from (2) and (5), we obtain

$$2\rho_1\cos\omega_1\sin^2\theta\cos\theta = \rho'\cos\omega'\sin\theta'(\sin\theta'\cos\theta - \sin\theta\cos\theta'),$$
$$= \rho'\cos\omega'\sin\theta'\sin(\theta'-\theta).$$

And eliminating $\rho\sin\omega$ from (3) and (4),

$$2\rho_1\sin\omega_1\sin^2\theta\cos\theta$$
$$= \rho'\sin\omega'\sin\theta'\{\sin\theta\cos\theta - \sin\theta'\cos\theta'\} - \rho'\sin^3\theta'\tan\epsilon;$$

and therefore

$$\tan\omega_1 = \tan\omega'\frac{\sin\theta\cos\theta - \sin\theta'\cos\theta'}{\sin(\theta'-\theta)} - \frac{\sin^2\theta'\tan\epsilon}{\cos\omega'\sin(\theta'-\theta)}$$
$$= \tan\omega'\cos(\theta+\theta') + \frac{\sin^2\theta'\tan\epsilon}{\cos\omega'\sin(\theta-\theta')},$$

which agrees with Art. 512 if $\epsilon = 0$.

Finally, we obtain from (2) and (5),

$$\rho_1\cos\omega_1 = \rho_0\cos\omega_0\frac{\sin(\theta'-\theta)}{\sin(\theta'+\theta)}.$$

Now these are the amplitudes of displacement perpendicular to the plane of incidence, and we see that they are connected by the same relation, as when both media are isotropic.

It will be observed that the flow of energy is at right angles to the magnetic and to the electromotive force as in Professor Poynting's theory, Art. 401.

532.] It appears therefore that certain phenomena of light can be explained on the electromagnetic hypothesis. The theory however in the form hitherto given fails to explain certain other phenomena, e. g. the rotation of the plane of polarisation, under the influence of magnetic force.

This investigation shews that on the usual hypothesis concerning the nature of dielectric displacement the magnetic force normal to the plane of the wave due to any system of periodic electromagnetic disturbances is zero.

As the system gives rise to no magnetic force in the normal to the wave, we should expect that any magnetic force in that direction due to external causes would have no influence on the system.

It is found, on the contrary, that in certain media a magnetic force in the direction of wave motion causes the plane of polarisation to rotate from left to right, as seen by an observer looking in the direction of wave motion. If we suppose the plane of polarisation to vary continuously, we still get on our hypothesis no magnetic force normal to the wave, and therefore cannot conclude that the normal force would cause the plane to rotate.

Maxwell gives an explanation of this phenomenon (*Magnetic Action on Light*, Chap. XXI) by resorting to a more general conception of dielectric displacement; instead of linear displacement he assumes two circular motions in opposite directions.

533.] Professor Rowland has also given an explanation of the phenomenon (Phil. Mag., April 1881) which is intimately connected with the electromagnetic theory of light.

It was first observed by E. H. Hall of Baltimore * that an electric current in a plane conductor under a magnetic force normal to the plane, if free to choose its course, is deflected across the lines of magnetic force. Professor Rowland assumes that the displacement currents in dielectric space have the same property. And this he interprets as an *electromotive force* whose components are

$$X = A\ (\beta \dot{h} - \gamma \dot{g})\ \&c.,$$

where A is a constant expressing the intensity of the force.

* Phil. Mag. April 1880.

Suppose then a wave of plane polarised light advancing in direction y, with a constant magnetic force β in that direction due to external causes. Then $\dot{g} = 0$, and $G = 0$. And making $\psi = 0$, as in preceding cases, we have for the total electromotive force the components

$$P = -\frac{dF}{dt} + A\beta\dot{h},$$

$$R = -\frac{dH}{dt} - A\beta\dot{f}.$$

Also $$f = \frac{K}{4\pi}P, \quad 4\pi\mu\dot{f} = -\nabla^2 F = -\frac{d^2F}{dy^2}\ \&c.,$$

whence we obtain

$$K\mu\left\{\frac{d^2F}{dt^2} - \frac{d}{dt}(A\beta\dot{h})\right\} - \frac{d^2F}{dy^2} = 0,$$

$$K\mu\left\{\frac{d^2H}{dt^2} + \frac{d}{dt}(A\beta\dot{f})\right\} - \frac{d^2H}{dy^2} = 0,$$

that is,

$$K\mu\left\{\frac{d^2F}{dt^2} + \frac{A\beta}{4\pi\mu}\frac{d}{dt}\frac{d^2H}{dy^2}\right\} - \frac{d^2F}{dy^2} = 0,$$

$$K\mu\left\{\frac{d^2H}{dt^2} - \frac{A\beta}{4\pi\mu}\frac{d}{dt}\frac{d^2F}{dy^2}\right\} - \frac{d^2H}{dy^2} = 0.$$

A solution of these equations is

$$F = r\cos\frac{2\pi}{\lambda}(vt - y)\cos mt,$$

$$H = r\cos\frac{2\pi}{\lambda}(vt - y)\sin mt,$$

where $$m = \frac{\pi A\beta}{\mu\lambda^2}$$

and $$v^2 = \frac{1}{K\mu} + \frac{A^2\beta^2}{16\mu^2\lambda^2},$$

giving a rotation of the plane of polarisation equal to $\frac{m}{v}$ per unit of distance traversed by the wave.

The reasoning seems open to objection thus. The new electromotive force is assumed to be proportional to the current. If we assume positive electricity e to be moving in the current with velocity v, the force on it will be proportional to ev. By the same reasoning, if negative electricity be moving in the opposite